Construction of A,D and Regular A-Optimal Spring Balance Designs using Orthogonal Arrays

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Abstract: From all possible block designs, in this paper deal to the construction of the spring balance weighing design based on the different orthogonal and covering array design and the estimation problem of individual weights of objects in A-optimal, D-optimal and regular A-optimal spring balance weighing design. We give the bounds of its value depending on whether the number of objects in experiment is odd or even. The theoretical considerations are illustrated with examples of construction of respective designs.

Keywords: Orthogonal Array Approach, Covering arrays, A-optimal design, D- optimal design, regular A-optimal spring balance design, Spring balance weighing design.

1. INTRODUCTION

1.1Orthogonal Array

An Orthogonal Array of strength t with N rows, k columns ($k \geq t$) and based on s symbols is an N x k array with entries 0,1,… , s -1, say, so that every N x t subarray contains each of the s^t possible t-tuples equally often as a row (say λ times) N must be a multiple of s^t , and $\lambda = N/s^t$ is the index of the array Notation: OA(N; k; s; t) or sometimes OA(N; s^k; t) (Cheng 1980; Beizer 1990).

1.2 Covering Array

A Covering array, denoted by CA_λ (N;t,p,v), represents an array of size N on v values, such that every N×t sub array contains all ordered subsets from the v values of size t at least λ times and p is the number of components (Yilmaz 2004; Colbourn 2008). OA is often too restrictive because it requires the parameters values to be uniform. Most of the time in practice, the values of the parameters are not uniform. To overcome these limitations, the covering array is emerged.

1.3 Spring balance design

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The basic principles of theory of the design of efficient experiments for estimating the true unknown weights of 'P' objects by means of a specified number of weighings n ($P \leq n$), and refers to the design of a certain class(Charyulu 2016).To weigh the objects either chemical balance or spring balance device are used. In spring balance only one pan is available for placing the objects. So the elements of the design matrix assumes only the values 1 and 0 according as the corresponding object is weighed in the combination or not.

The results of n weighing operations determining the individual weights of P objects fit into the linear model i.e. we determine unknown measurements of p objects using n operations according to the linear model

$$y = X w + e \qquad (1.1)$$

The model is consideration has been extensively analyzed from the perspective of optimal design, and all Φ_p- optimal design are known.

where y is an n×1 random vector of the observations. The design matrix $X = (x_{ij})$ usually called weighing matrix belongs to the class $\Phi_{n\times p}(0, 1)$, which denotes the class of n×p matrices of known elements $x_{ij} = 0$ or 1 according as in the i th weighing operation the j th object is not placed on the pan or is placed. w is a p×1 vector of unknown weights of objects and e is an n×1 random vector of errors. We assume, that there are no systematic errors, the variances of errors are not equal and the errors are uncorrelated, i.e. $E(e) = 0_n$ and $Var(e) = \sigma^2 G$, where 0_n denotes the n×1 vector with zero elements everywhere, G is the known n×n diagonal positive definite matrix(Charyulu 2016; Banerjee1949).

Any spring balance weighing design is said to be singular or nonsingular, depending on whether the matrix $X'G^{-1}X$ is singular or nonsingular, respectively. It is obvious, that if G is the known positive definite matrix then the matrix $X'G^{-1}X$ is nonsingular if and only if the matrix $X'X$ is nonsingular, i.e. if and only if X is full column rank $r(X) = p$. However, if $X'^{G-1}X$ is nonsingular, then the generalized least squares estimator of w is given by $\hat{w} = (X'G^{-1}X)^{-1}X'G^{-1}y$ and the variance matrix of $\hat{w}$ is $Var(\hat{w}) = \sigma^2 (X'G^{-1}X)^{-1}$. If X'X is singular take 't' additional weighing's ($t > 0$) made with all the 'P' objects being placed in the pan (Banerjee1949).

Several authors using several techniques for the construction and estimation of weights of objects using spring balance design(Banerjee1949; Meena 1980; Jacroux 1983).

A design is said to be optimal within a given class provided it is determined to be "best" by some optimality function Φ. For given variance matrix of the errors $\sigma^2 G$, the A-optimal design is the design X for which, the sum of variances of estimators for unknown parameters is minimal, i.e. $tr(X'G-1X)^{-1}$ is minimal in $\Phi_{n\times p}(0, 1)$. Moreover, the design for which the sum of variances ofestimators for parameters attains the lowest bound in $\Phi_{n\times p}(0, 1)$ is called the regular A-optimal design(Jacroux 1983).

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The main purpose of this paper is to obtain optimality results of some spring balance weighing design which canconstructed byorthogonal and covering array design. In the section 2 we construct spring balance design with orthogonal design and then obtain A, D and Regular A-optimal design.In the section 3 we construct spring balance weighing design with covering array design and then obtain A, D and Regular A- optimal design.

1.4 Optimality Criteria

One of the first to state a criterion and obtain optimal designs for regression problems was Smith (1918). The criterion she proposed was: minimize the maximum variance of any predicted value (obtained by using the regression function) over the experimental space.I.e.

$\min_{xi,i=1,\ldots,n} \max_{x\in X} Var(\hat{y}_x)$.

A second criterion, proposed by Wald (1943), puts the emphasis on the quality of the parameter estimates. The criterion is to maximize the determinant of X' X. This was called D-optimality by Kiefer and Wolfowitz (1959).

$\max_{xi,i=1,\ldots,n} |X' X|$.

While these criteria are the ones which have received the most attention minimizes the average variance of the parameter estimates (Chernoff, 1953), so called A-optimality.

$\min_{xi,i=1,\ldots,n} trace(X' X^{-1})$,

why we use D-optimal designs? When we have a limited budget, time or cost and cannot run a completely replicated factorial design. For example, suppose we want to study the response to three factors: A with four levels, B with five levels, and C with six levels. One complete replication of this experiment would require 4 x 5 x 6 = 120 points. Suppose we can afford only 20 points. Which 20 of the 120 possible should we use? The D-optimal design provides a reasonable choice.

Let D be a class of weighing designs for estimating the weights of 'p' objects(Charyulu 2016).A design D is said to be optimal if $\emptyset[M(d)] \leq \emptyset[M(d^*)]$ for any $d^* \in D$. The function f is said to be the criterion function. Then the following optimality criteria are defined for comparing designs belongs to D.

A-Optimality: A design d* belonging to D is said to be A-optimal in D if Trace $(X'X)^{-1})d^* \leq$Trace $(X'X)^{-1})d$ for any other design *d* belonging to D.

D-Optimality: A design d* belonging to D is said to be D-optimal in D if $|(X'X)^{-1}|d^* \leq |(X'X)^{-1}|d$ for any other design d belonging to D.

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1.5 Regular A-optimal Design

For any experimental setting, i.e. for fixed n, p and G, there is always a number of designs available for using. In each class of available designs, the regular A-optimal design is considered. Furthermore, the main difficulty in carrying out the construction is that each form of G requires the specific investigations(Graczyk 2012).That's why we consider the experimental situation we determine unknown measurements of p objects in n = $\sum_{i=1}^{h} n_s$ measurement operations under model 1.1. It is assumed that n_s measurements are taken in different h conditions or at different h installations. So, the variance matrix of errors $\sigma^2 G$ is given by the matrix

$$G=\begin{bmatrix} g_1^{-1} I_{n_1} & O_{n_1} O'_{n_2} \cdots & O_{n_1} O'_{n_h} \\ O_{n_1} O'_{n_1} & g_2^{-1} I_{n_1} \cdots & O_{n_2} O'_{n_2} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ O_{n_h} O'_{n_2} & O_{n_h} O'_{n_2} \cdots & g_h^{-1} I_{n_h} \end{bmatrix} \quad (1.2)$$

where$g_s > 0$ denotes the factor of precision, s = 1, 2, ..., h. Consequently, according to the form of G we write the design matrix $X \in \Phi_{n\times p}(0, 1)$ as

$$X=\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \cdots . \\ X_h \end{bmatrix} (1.3)$$

WhereXs is $n_{s\times p}$ design matrix of any spring balance weighing design.

Theorem 1.1 Any $X \in \Phi_{n\times p}(0, 1)$ in (1.3) with the variance matrix of errors $\sigma^2 G$ is the regular A-optimal spring balance weighing design presented by(Jacroux 1983) if and only if

A) for even p, $X'G^{-1}X = p/\ 4(p-1)\ tr(G^{-1})\ I_p + p-2/\ 4(p-1)\ tr(G^{-1})1_p 1'_p$, (1.4)
Or
B) for odd p, $X'G-1X = p+1\ /4p\ tr(G^{-1})\ (Ip + 1_p 1'_p)$.

If $X \in \Phi_{n\times p}(0, 1)$ satisfies the equalities given in Theorem 1.1 then X is the regular A-optimal design for any G in (1.2).Hence X is the regular A-optimal design in the special case when $G = I_n$ and

$$tr(X'X)^{-1} \geq \begin{cases} \frac{4(p^2-2p+2)}{np}, & \text{if p is even} \\ \frac{4p^3}{n(p+1)^2}, & \text{if p is odd} \end{cases} \quad (1.5)$$

2. CONSTRUCTION OF SPRING BALANCE DESIGN USING ORTHOGONAL ARRAY DESIGN

In this section, Consider a orthogonal design of two-dimensional array table, or matrix of N rows and k columns with parameters N, k, s, t and λ such that OA(N; k; s; t) , an attempt is made to propose methods for the construction of spring balance weighing designs using orthogonal design. The method is illustrated through suitable examples.

Theorem 1.1:Let N be even order in a orthogonal design with parameter N=2(2t-2) ; k=2t-2; S=2 (t-2); ; λ=t-2 where t=1,2,3… .Suppose that N treatments are associated with each objects of K .The treatments associated with an objects a of k are denoted by a_1, $a_{2,}$, a_3….. a_n. Suppose that it is possible to find a set of p blocks b_1, b_2, b_3…. $b_p$which satisfy the following conditions.

(i) Each block contains k treatments
(ii)K blocks contain different parameters, the difference arising from p blocks.
(iii)S is the number of possible variables in the OA design.
(iv) t is the number of variables in each N×t sub arrays.
(v) λ represents the number of times each combination of values appear in each sub array.

then consider any x$\in \emptyset_{n*k}$$I_k$is the identity matrix of order K. Derive the matrix X by adding the i^{th} row of I_k to those row of n_{ij} having 0(zero) in the j^{th} column. Then the resulting n_{ij}^* constitutes a spring balance design with K objects.

Case :1 :-Let N be even in a orthogonal design with parameter N=2(2t-2) ; k=2t-2; S=2 (t-2); ; λ=t-2 where t=1,2,3… then any x$\in \emptyset_{n*k}$$I_k$is the identity matrix of order K. Derive the matrix X by adding the i^{th} row of I_k to those row of n_{ij} having 0(zero) in the j^{th} column. Then the resulting n_{ij}^* constitutes a spring balance design with K objects.

Example: OA(8,4,2,3) N is equal to 8 represent different treatments to be run, K columns represents different parameters, S represents number of possible variables ,t represents the number of variables in each N×t sub arrays and λ represents the number of times each combination of values appear in each sub array can be calculated by $\lambda=N/s^t$

Consider orthogonal design OA(8,4,2,3) and I_k, where $I_k$is the identity matrix of order K

OA(8,4,2,3)			
0	0	0	0
0	0	1	1
0	1	0	1

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0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Derive the matrix X by adding the i^{th} row of I_k to those row of n_{ij} having 0(zero) in the j^{th} column. Then the resulting n_{ij}^{*} constitutes a spring balance design with K objects. Each objects is weighted twice and the average weight is used on the estimate of the weight of that object.

Spring Balance Design X			
1	0	1	1
1	1	0	1
1	1	1	0
0	1	1	1
1	1	0	1
1	1	1	0
0	1	1	1
1	0	1	1
1	1	1	0
0	1	1	1
1	0	1	1
1	1	0	1

First calculate variance and efficiency of the orthogonal design.The matrix XX' in OA (8,4,2,3) will take in the form

$$\begin{pmatrix} 4 & 2 & 2 & 2 \\ 2 & 4 & 2 & 2 \\ 2 & 2 & 4 & 2 \\ 2 & 2 & 2 & 4 \end{pmatrix}$$

The variance of the estimated weight of each of the p object for such a design can be easily seen to be

$$\frac{r+\lambda(p-2)}{(r-\lambda)\{r+\lambda(p-1)\}}\sigma^2 \quad \text{for zero bias} \rightarrow \frac{2}{5}\sigma^2$$

Where P is the number of objects to be weighted and r and λ have meanings similar to those in connections with balanced incomplete block designs i.e. r is the no. of times each objectweighted and λ is the no. of times each pair of objects is weighted together under such design may however be kept as the standard with which the efficiency of a given design may be calculated, the efficiency of the above design will therefore for zero bias be

$$\frac{(r-\lambda)\{r+\lambda(p-1)\}}{N\{r+\lambda(p-2)\}} \to \frac{1}{3}$$

Let us consider $X \in \Phi_{12\times4}(0, 1)$ we construct the spring balance design with orthogonal design with parameter OA(8,4,2,3).Next we check optimality criterion of spring balance design in case :1.

A-Optimality: A design d* belonging to D is said to be A-optimal in D if Trace $(X'X)^{-1}$)d* ≤Trace $(X'X)^{-1}$)d for any other design *d* belonging to D.

D-Optimality: A design d* belonging to D is said to be D-optimal in D if $|(X'X)^{-1}|$d* ≤ $|(X'X)^{-1}|$d for any other design d belonging to D.

Regular A-optimal design

If $X \in \Phi_{n\times p}(0, 1)$ satisfies the equalities(if p is even) given in Theorem 1.1 then X is the regular A-optimal design for any G in (1.2). Hence X is the regular A-optimal design in the special case when $G = I_n$ and

$$tr(X'X)^{-1} \geq \begin{cases} \frac{4(p^2-2p+2)}{np}, & \text{if p is even} \\ \frac{4p^3}{n(p+1)^2}, & \text{if p is odd} \end{cases}$$

Case :2 :- Let N be even in a orthogonal design with parameter N=2(2t-2) ; k=2(2t)-1; S=2 t-2; λ=t-1 where t=1,2,3… then any $X \in \Phi_{n\times p}(0, 1)I_k$is the identity matrix of order K. Derive the matrix X by adding the i^{th} row of I_k to those row of n_{ij} having 0(zero) in the j^{th} column. Then the resulting n_{ij}^* constitutes a spring balance design with K objects.

OA(8,7,2,2)						
0	0	0	0	0	0	0
0	0	0	1	1	1	1
0	1	1	0	0	1	1
0	1	1	1	1	0	0
1	0	1	0	1	0	1
1	0	1	1	0	1	0
1	1	0	0	1	1	0
1	1	0	1	0	0	1

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Consider orthogonal design OA(8,7,2,2) and I_k, where I_k is the identity matrix of order K. Derive the matrix X by adding the i^{th} row of I_k to those row of n_{ij} having 0(zero) in the j^{th} column. Then the resulting n_{ij}^* constitutes a spring balance design with K objects.

Spring Balance Design X						
1	0	0	1	1	1	1
1	1	1	0	0	1	1
1	1	1	1	1	0	0
0	1	0	1	1	1	1
1	1	1	0	1	0	1
1	1	1	1	0	1	0
0	0	1	1	1	1	1
1	1	1	0	1	1	0
1	1	1	1	0	0	1
0	1	1	1	0	1	1
1	0	1	1	1	0	1
1	1	0	1	1	1	0
0	1	1	0	1	1	1
1	0	1	1	1	1	0
1	1	0	1	1	0	1
0	1	1	1	1	1	0
1	0	1	0	1	1	1
1	1	0	1	0	1	1
0	1	1	1	1	0	1
1	0	1	1	0	1	1
1	1	0	0	1	1	1

Let us consider $X \in \Phi_{21\times7}(0, 1)$ we construct the spring balance design with orthogonal design with parameter OA (8,7,2,2).
Next we check optimality criterion of spring balance design in case :2:-

A-Optimality: A design d* belonging to D is said to be A-optimal in D if Trace $(X'X)^{-1}$)d* ≤Trace $(X'X)^{-1}$)d for any other design *d* belonging to D.

D-Optimality: A design d* belonging to D is said to be D-optimal in D if $|(X'X)^{-1}|$d* ≤ $|(X'X)^{-1}|$d for any other design d belonging to D.

Regular A-optimal design

If $X \in \Phi_{n \times p}(0, 1)$ satisfies the equalities(if p is odd) given in Theorem 1.1 then X is the regular A-optimal design for any G in (1.2). Hence X is the regular A-optimal design in the special case when $G = I_n$ and

$$tr(X'X)^{-1} \geq \begin{cases} \frac{4(p^2-2p+2)}{np}, & \text{if p is even} \\ \frac{4p^3}{n(p+1)^2}, & \text{if p is odd} \end{cases}$$

3. CONSTRUCTION OF SPRING BALANCE DESIGN USING COVERING ARRAY DESIGN

It is desirable for any construction method to construct CAs to cover all the required t-interactions with a minimum number of rows. Generally, CAs are constructed computationally to ensure minimization in terms of size and time.

Now if each pair doesn't have to occur the same number of times but the same 7 parameters having two values each this covering array can be created.CA is testing all combinations of two parameters but doing it in only 6 tests not 8.

Case :3 Let N be even in a orthogonal design with parameter N=2(2t-1) ; p=2(2t)-1; v=2 t-2 where t=1,2,3… then any $X \in \Phi_{n \times p}(0, 1)$ $I_p$is the identity matrix of order p. Derive the matrix X by adding the i^{th} row of I_p to those row of n_{ij} having 0(zero) in the j^{th} column. Then the resulting n_{ij}^* constitutes a spring balance design with p objects.

CA(6,7,2)						
0	0	0	0	0	0	0
0	0	1	1	1	0	1
0	1	0	1	0	1	1
1	0	0	0	1	1	1
1	1	1	0	0	0	1
1	1	1	1	1	1	0

Consider covering array design CA(6,7,2) and I_p, where $I_p$is the identity matrix of order p. Derive the matrix X by adding the i^{th} row of I_p to those row of n_{ij} having 0(zero) in the j^{th} column. Then the resulting n_{ij}^* constitutes a spring balance design with p objects.

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Spring Balance Design X					
1	0	1	1	1	0
1	1	0	1	0	1
0	1	1	1	1	0
1	1	0	0	1	1
0	1	1	1	0	1
1	0	1	0	1	1
1	0	0	1	1	1
1	1	1	1	0	0
0	1	0	1	1	1
1	1	1	0	1	0
0	0	1	1	1	1
1	1	1	0	0	1

Let us consider $X \in \Phi_{12\times 7}(0, 1)$ we construct the spring balance design with orthogonal design with parameter CA (6,7, 2,).
Next we check optimality criterion of spring balance design in case: 2:-

A-Optimality: A design d* belonging to D is said to be A-optimal in D if Trace $(X'X)^{-1}$)d* ≤Trace $(X'X)^{-1}$)d for any other design *d* belonging to D.

D-Optimality: A design d* belonging to D is said to be D-optimal in D if $|(X'X)^{-1}|$d* ≤ $|(X'X)^{-1}|$d for any other design d belonging to D.

Regular A-optimal design

If $X \in \Phi_{n\times p}(0, 1)$ satisfies the equalities (if p is odd) given in Theorem 1.1 then X is the regular A-optimal design for any G in (1.2). Hence X is the regular A-optimal design in the special case when $G = I_n$ and

$$tr(X'X)^{-1} \geq \begin{cases} \frac{4(p^2-2p+2)}{np}, & \text{if p is even} \\ \frac{4p^3}{n(p+1)^2}, & \text{if p is odd} \end{cases}$$

4.DISCUSSION

The optimal design approach is a powerful and flexible way to generate efficient experimental designs. It allows computing designs with any number of observations and experimental variables (qualitative or quantitative variables) for fitting linear or nonlinear model in the unknown parameters. The study of present paper, to understand the methods of construction of spring balance design with orthogonal arrays design and covering array design and apply optimality criterion such that A-optimal, D-optimal design and Regular A-Optimal in the class $(0,1)\Phi_{n\times p}$ based on the design matrix, the efficiency of the D-optimal design is more than 50% with respect to any orthogonal invariant criterion and for any dimension of the model. We aimed to demonstrate the difference between OA and CA and report their application by software testing. Covering arrays work extremely well in software testing and are much easier to create then orthogonal arrays.

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