STABILITY ANALYSIS OF A PREY PREDATOR AND AMMENSAL

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Abstract:
We describe a mathematical model for the interaction between three species ecological model consists of first (x), second (y) and third species (z). The first species is preying on second, second species is ammensal on the third. A time delay is included in the interaction first and second species. The system is explained by couple of delay differential equations. The co-existing state is identified and also characterizes the local stability analysis at this state. Parametric Hopf bifurcation, instability nature is identified and supported with a suitable numerical simulation using MATLAB.

Keywords: Prey, Predator Co-existing state, local stability, Hopf bifurcation
AMS classification: 34 DXX

1. Introduction

Differential equations are most popular in explaining the mathematical models I ecology. The stability analysis concept is explained in detail by Braun [9] and Simon's [10]. The ecological models are initiated by studied Lokta [1] and Volterra [2]. The Mathematical models and its stability analysis discussed by Kapur [3, 4]. qualitative analysis plays a big role in analysing these models due to the difficulty in finding analytical solutions due to the non-linearity of the models arise ecology. The qualitative analyses of ecological models are widely studied by authors [5-7]. The stability of analysis of delay-differential equations are significant in ecology. The time delays are influence the dynamics of the system and tend to destabilize or stabilizes the system. The systems with delay arguments and the qualitative analysis are widely discussed by the authors [11-13]. The nature of the delay argument cause unbounded growth and extinction of populations leads to instability tendency of models. The delay argument may classify in to continuous, discrete, distributed etc. The time lags can be discrete or continuous. These lags will change the stable equilibrium to unstable or vice-versa. The delay models in population dynamics are widely studied by paparao [14-21]. In this paper we take a logistic growth model of three species for investigation. In this model first species is preying on second and second is ammensal to third species. A discrete time lag is incorporated in the interaction of first and second species. The model is studied by a couple of delay-differential equations. The co-existing equilibrium point is identified and discussed the dynamics at this point. Numerical simulation is carried out carried out in support of stability analysis. It is shown that the system exhibits instability trendies leads to Hopf bifurcation.

2. Model Equations:

The proposed ecological system can be modelled into the following system of equations given by

$$\frac{dx}{dt} = a_1 x\left(1 - \frac{x}{k_1}\right) + a_{12}x(t-\tau)y(t-\tau)$$

$$\frac{dy}{dt} = a_2 y\left(1 - \frac{y}{k_2}\right) - a_{21}x(t-\tau)y(t-\tau) \quad (2.1)$$

$$\frac{dz}{dt} = a_3 z\left(1 - \frac{z}{k_3}\right) - a_{32}yz$$

2.1 Nomenclature:

S.No.	Parameter	Description
1	x,y,z	Populations of three species
2	a_i	Natural growth rates of three species

4	α_{12}	Interaction rate of first and second species (positive value)
5	α_{21}	Interaction rate of second and first species (negative value)
6	α_{32}	Interaction rate of third (ammensal) and second species (negative value)
7	k_1, k_2, k_3	Carrying capacities of first, second and third species populations respectively.

3.Equilibrium Points

Equating the system of equations (2.1) to zero, and derive the co-existing state is given by

$$E(\overline{x},\overline{y},\overline{z}) = \left(\begin{array}{c} \frac{k_1(a_1a_2+a_{12}a_2k_2)}{(a_1a_2+a_{12}a_{21}k_1k_2)}, \\ \frac{k_2(a_1a_2-a_{21}a_1k_1)}{(a_1a_2+a_{12}a_{21}k_1k_2)}, \\ \frac{k_3\{(a_3a_1a_2+a_{12}a_{21}k_1k_2)-k_2\alpha_{32}(a_1a_2+a_{21}a_1k_1)\}}{a_3(a_1a_2+a_{12}a_{21}k_1k_2)} \end{array} \right) \tag{3.1}$$

Co-existing state exist if (i) $a_2a_2 > a_1\alpha_{21}k_1$
(ii) $(a_3a_1a_2 + a_{12}a_{21}k_1k_2) > k_2\alpha_{32}(a_1a_2 + a_{21}a_1k_1)$ are Satisfied (3.2)

4. Local Stability at Co-existing State

Theorem1: The co-existing state is locally asymptotically stable

Proof: The variational matrix for the system (2.1) is

$$J = \begin{bmatrix} a_1 - \frac{2a_1\overline{x}}{k_1} + a_{12}\overline{y}e^{-\lambda\tau} & a_{12}\overline{x}e^{-\lambda\tau} & 0 \\ -a_{21}\overline{y}e^{-\lambda\tau} & a_2 - \frac{2a_2\overline{y}}{k_2} + a_{12}\overline{y}e^{-\lambda\tau} & 0 \\ 0 & -a_{32}\overline{z} & a_3 - \frac{2a_3\overline{z}}{k_3} - a_{32}\overline{y} \end{bmatrix} \tag{4.1}$$

Characteristic equation of the (4.1) is given by

$$\psi(\lambda,\tau) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 + e^{-\lambda\tau}(q_1\lambda^2 + q_2\lambda + q_3) = 0 \tag{4.2}$$

$$P_1 = \frac{2a_1\overline{x}}{k_1} + \frac{2a_2\overline{y}}{k_2} + \frac{2a_3\overline{z}}{k_3} - (a_1 + a_2 + a_3)$$

$$P_2 = a_1a_2 + a_1a_3 + a_2a_3 + \frac{4a_1a_2\overline{x}\overline{y}}{k_1k_2} + \frac{4a_1a_3\overline{x}\overline{z}}{k_1k_3} + \frac{4a_2a_3\overline{y}\overline{z}}{k_2k_2} + \frac{2a_1a_{32}\overline{x}\overline{y}}{k_1} + \frac{2a_2a_{32}\overline{y}^2}{k_2}$$
$$-\frac{2a_1a_2\overline{y}}{k_2} - \frac{2a_1a_2\overline{x}}{k_1} - \frac{2a_1a_3\overline{z}}{k_3} - \frac{2a_2a_3\overline{z}}{k_3} - a_{32}a_1\overline{y} - a_{32}a_2\overline{y} - \frac{2a_1a_3\overline{x}}{k_1} - \frac{2a_3a_2\overline{y}}{k_2}$$

$$P_3 = \frac{2a_1a_2a_3\overline{y}}{k_2} + \frac{2a_1a_2a_3\overline{x}}{k_1} + \frac{2a_1a_2a_3\overline{z}}{k_3} + \frac{4a_1a_2a_{32}\overline{x}\overline{y}^2}{k_1k_2} + \frac{8a_1a_2a_3\overline{x}\overline{y}\overline{z}}{k_1k_2k_3} + a_1a_2a_{32}\overline{y}$$
$$-\frac{4a_1a_2a_3\overline{y}\overline{z}}{k_2} - \frac{4a_1a_2a_3\overline{z}\overline{x}}{k_3} - \frac{2a_1a_3a_{32}\overline{y}^2}{k_2} - \frac{2a_1a_2a_{32}\overline{x}\overline{y}}{k_1} - a_1a_2a_3 - \frac{4a_1a_2a_3\overline{x}\overline{y}}{k_1k_2}$$

$$q_1 = a_{21}\overline{x} - a_{12}\overline{y}$$

$$q_2 = a_1a_{12}\bar{x} + a_2a_{12}\bar{y} + a_{12}a_3\bar{y} + a_{21}a_{32}\bar{x}\bar{y} + \frac{2a_3a_{21}\bar{x}\bar{z}}{k_3} - \frac{2a_2a_{12}\bar{y}^2}{k_2} - a_{12}a_{32}\bar{y}^2 - a_3a_{21}\bar{x} - a_1a_{21}\bar{x}$$

$$q_3 = a_1a_{21}a_3\bar{x} + \frac{2a_1a_{21}a_{32}\bar{x}\bar{y}}{k_1} + \frac{2a_{12}a_3a_2\bar{y}^2}{k_2} + 2a_{12}a_{32}a_2\bar{y}^2 + \frac{4a_1a_{21}a_3\bar{x}\bar{z}}{k_1k_3} + \frac{2a_2a_{12}a_3\bar{y}\bar{z}}{k_3}$$

$$-\frac{2a_1a_3a_{21}\bar{x}\bar{z}}{k_3} - \frac{4a_2a_3a_{12}\bar{y}^2}{k_2} - \frac{2a_2a_{32}a_{12}\bar{y}^3}{k_2} - \frac{2a_{12}a_3\bar{y}\bar{z}}{k_3} - a_{21}a_{32}\bar{x}\bar{y} - \frac{2a_1a_{21}a_3\bar{x}}{k_1} - a_{12}a_2a_3\bar{y}$$

Which can be written as $\psi(\lambda,\tau) = P(\omega) + Q(\omega)e^{-\tau\omega}$

Case (i) For $\tau = 0$

The characteristic equation obtained from (4.2) by putting $\tau = 0$ given by the following equation

$$\psi(\lambda,0) = -\left(\frac{a_3\bar{z}}{k_3} + \lambda\right)\left[\lambda^2 + \lambda\left(\frac{a_1\bar{x}}{k_1} + \frac{a_2\bar{y}}{k_2}\right) + \left(\frac{a_1a_2\bar{x}\bar{y}}{k_1k_2} + a_{12}a_{21}\bar{x}\bar{y}\right)\right] = 0$$

$$\left(\frac{a_3\bar{z}}{k_3} + \lambda\right) = 0 \text{ or } \left[\lambda^2 + \lambda\left(\frac{a_1\bar{x}}{k_1} + \frac{a_2\bar{y}}{k_2}\right) + \left(\frac{a_1a_2\bar{x}\bar{y}}{k_1k_2} + a_{12}a_{21}\bar{x}\bar{y}\right)\right] = 0$$

$$\lambda = -\frac{a_3\bar{z}}{k_3} = 0$$

$$and\left[\lambda^2 + \lambda\left(\frac{a_1\bar{x}}{k_1} + \frac{a_2\bar{y}}{k_2}\right) + \left(\frac{a_1a_2\overline{xy}}{k_1k_2} + a_{12}a_{21}\overline{xy}\right)\right] = 0 \qquad (4.3)$$

One of the roots is negative i.e., $-\frac{a_3\bar{z}}{k_3}$

From the equation (3.B.3.3) find the remaining two roots. if the two roots have negative real roots if the trace of the equation $\left(\frac{-b}{a}\right)$ is negative and the determinant $\left(\frac{c}{a}\right)$ is positive.

The trace and determinant from the equation (3.B.3.3) are given as follows

$$\text{Here the trace is} = \frac{-b}{a} = \frac{-\left(a_1\bar{x}k_2 + a_2\bar{y}k_1\right)}{k_1k_2} < 0$$

$$\text{Determinant} = \frac{c}{a} = \frac{\left(a_1a_2 + a_{12}a_{21}k_1k_2\right)\bar{x}\bar{y}}{k_1k_2} > 0$$

Therefore, the system (2.1) is locally asymptotically stable at co-existing state.

Therefore, the co-existing state is locally asymptotically stable.

Case (ii) Let $\tau > 0$**:** Suppose there is a positive τ_0 such that the equation (4.2) has pair of purely imaginary root, let the roots be $\pm i\omega, \omega > 0,$ therefore $i\omega$ satisfies the equation (4.2)

$$(i\omega)^3 + p_1(i\omega)^2 + p_2(i\omega) + p_3 + e^{-i\omega\tau}(q_1(i\omega)^2 + q_2(i\omega) + q_3) = 0$$

$$-\omega^2p_1 + p_3 - q_1\omega^2\cos\omega\tau + q_3\cos\omega\tau + q_2\omega\sin\omega\tau +$$

$$i[-w^3 + wp_2 + q_2\omega\cos\omega\tau + q_1\omega^2\sin\omega\tau - q_3\sin\omega\tau] = 0$$

Separating real and imaginary parts, we get

$$(q_3 - q_1\omega^2)\cos\omega\tau + q_2\omega\sin\omega\tau = \omega^2p_1 - p_3 \qquad (4.4)$$

$$q_2\omega\cos\omega\tau - (q_3 - q_1\omega^2)\sin\omega\tau = \omega^3 - \omega p_2 \qquad (4.5)$$

On adding, the two equations after squaring, we get

From the above equations we get the following equation (by squaring and add the two results)

$$(q_3 - q_1\omega)^2 + (q_2\omega)^2 = (\omega^2p_1 - p_3)^2 + (\omega^3 - \omega p_2)^2$$

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$$\omega^6+\omega^4(p_1^{\,2}-2p_2-q_1^{\,2})+\omega^2(p_2^2-2p_1p_3-q_2^2+2q_1q_3)+q_3^{\,2}+p_3^{\,2}=0$$

Let $\psi(p)=p^3+p^2N_1+pN_2+N_3=0$ (4.6)

Where

$N_1=p_1^2-2p_1-q_1^{\,2}$

$N_2=p_2^2-2p_1p_2-q_2^2+2q_1q_3$

$N_3=q_3^3+p_3^3$

$p=\omega^2$

$\therefore \psi(p)=o$

If we assume that $N_1>0, N_2>0, N_3>0$ then equation (4.2) has no positive real roots. Therefore, the equation (4.2) admits negative real roots. Hence, we can derive the conditions for existences of stability at equilibrium point.

Theorem 4.1 The system (2.1) is locally asymptotically stable at co-existing state for all τ , if the following conditions hold.

(i). $(p_1+q_1)>0,\ (p_2+q_2)>0$, $(p_3+q_3)>0$

(ii). $N_1>0, N_2>0, N_3>0$

Proof: Any one of N_1, N_2, N_3 is negative. Then equation (4.2) has a positive root ω_0

Eliminating $\cos\omega\lambda$, from the equations (4.4) & (4.5), we have

$$\cos\omega\tau=\frac{\begin{vmatrix}\omega^2p_1-p_3 & q_2\omega\\ \omega^3-\omega p_2 & -(q_3-q_1\omega^2)\end{vmatrix}}{\begin{vmatrix}q_3-q_1\omega^2 & q_2\omega\\ q_2\omega & -(q_3-q_1\omega^2)\end{vmatrix}}$$

($\because$ by using Cramer's rule in determinants)

$$\cos\omega\tau=\frac{\omega^2p_1q_3-p_3q_3-\omega^4p_1q_1+\omega^2q_1p_3+q_3w^4-p_2q_3\omega^2}{q_3^2+q_1^2\omega^4-2q_1q_3w^2+q_2^2\omega^2}$$

$$\tau_k=\frac{1}{\omega_0}\cos^{-1}\left[\frac{\omega_0^4(q_3-p_1q_1)+\omega_0^2(p_1q_3+q_1p_3-p_2q_3)-p_3q_3}{q_1^2\omega_0^4+\omega_0^2(q_2^2-2q_1q_3)+q_3^2}\right]+\frac{2k\pi}{\omega_0}$$

where $k=0,1,2,3$...............

5. HOPF BIFURCATION

Theorem 5.1: The sufficient condition for the system (4.1) admits bifurcation along the co-existing state E if $\tau>\tau_0$ and locally asymptotically stable If $0<\tau<\tau_0$

Proof: Hopf bifurcation occurs when the real part of $\lambda(t)$ become positive when $\tau>\tau_0$ and the steady state become unstable moreover, when τ passes through the critical value τ_0 .

To check this result, differentiating the equation (4.2) With respect to τ, we get

$$3\lambda^2\frac{d\lambda}{d\tau}+2p_1\lambda\frac{d\lambda}{d\tau}+p_2\frac{d\lambda}{d\tau}+e^{-\lambda\tau}(2q_1\lambda\frac{d\lambda}{d\tau}+q_2\frac{d\lambda}{d\tau})+(q_1\lambda^2+q_2\lambda+q_3)(-\lambda-\lambda\frac{d\lambda}{d\tau})e^{-\lambda\tau}=0$$

$$\frac{d\lambda}{d\tau}\left[3\lambda^2+2p_1\lambda+p_2+e^{-\lambda\tau}(2q_1\lambda+q_2)-(q_1\lambda^2+q_2\lambda+q_3)\lambda e^{-\lambda\tau}\right]=(q_1\lambda^2+q_2\lambda+q_3)\lambda e^{-\lambda\tau}$$

$$\left[\frac{d\lambda}{d\tau}\right]^{-1}=\frac{\left[3\lambda^2+2p_1\lambda+p_2+e^{-\lambda\tau}(2q_1\lambda+q_2)-(q_1\lambda^2+q_2\lambda+q_3)\pi e^{-\lambda\tau}\right]}{(q_1\lambda^2+q_2\lambda+q_3)\lambda e^{-\lambda\tau}}$$

$$\left[\frac{d\lambda}{d\tau}\right]^{-1}=\frac{3\lambda^2+2p_1\lambda+p_2}{(q_1\lambda^2+q_2\lambda+q_3)\lambda e^{-\lambda\tau}}+\frac{(2q_1\lambda+q_2)}{(q_1\lambda^2+q_2\lambda+q_3)\lambda}-\frac{\tau}{\lambda}$$

$$\left[\frac{d\lambda}{d\tau}\right]^{-1}=\frac{3\lambda^2+2p_1\lambda+p_2}{-\lambda(\lambda^3+p_1\lambda^2+p_2\lambda+p_3)}+\frac{(2q_1\lambda+q_2)}{(q_1\lambda^2+q_2\lambda+q_3)\lambda}-\frac{\tau}{\lambda}$$

$\lambda=i\omega$

$$\left[\frac{d\lambda}{d\tau}\right]^{-1}=\frac{1}{\omega_0}\left[\frac{-3\omega_0{}^2+2ip_1\omega_0+p_2}{(-\omega_0{}^3+p_2\omega_0+i(p_1\omega_0^2-p_3)^2}+\frac{(2iq_1\omega_0+q_2)}{-q_2\omega_0+i(q_3-q_1\omega_0^2)}+\tau i\right]$$

$$\left[\frac{d\lambda}{d\tau}\right]^{-1}=\frac{1}{\omega_0}\left[\frac{(-3\omega_0{}^2+2ip_1\omega_0+p_2)((-\omega_0{}^3+p_2\omega_0-i(p_1\omega_0^2-p_3)}{(-\omega_0{}^3+p_2\omega_0)^2+(p_1\omega_0^2-p_3)^2}+\frac{(2iq_1\omega_0+q_2)(-q_2\omega_0-i(q_3-q_1\omega_0^2)}{(q_2\omega_0)^2+(q_3-q_1\omega_0^2)^2}+\tau i\right]$$

Real part of

$$\left[\frac{d\lambda}{d\tau}\right]^{-1}=\frac{1}{\omega_0}\left[\frac{(-3\omega_0{}^2+p_2)(-\omega_0{}^3+p_2\omega_0)+2p_1\omega_0(p_1\omega_0^2-p_3)}{(-\omega_0{}^3+p_2\omega_0)^2+(p_1\omega_0^2-p_3)^2}+\frac{-q_2{}^2\omega_0+2q_1\omega_0(q_3-q_1\omega_0^2)}{(q_2\omega_0)^2+(q_3-q_1\omega_0^2)^2}\right]$$

$$(-\omega_0{}^3+p_2\omega_0)^2+(p_1\omega_0^2-p_3)^2=(q_2\omega_0)^2+(q_3-q_1\omega_0^2)^2$$

$$=\frac{1}{\omega_0}\left[\frac{3\omega_0^5+\omega_0^3(2p_1^2-p_2-3p_2-2q_1^2)+(p_2^2-2p_1p_3+2q_1q_2-q_2^2)\omega_0}{(q_2\omega_0)^2+(q_3-q_1\omega_0^2)^2}\right]$$

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}=\left[\frac{3\omega_0^4+\omega_0^2(2p_1^2-4p_2-2q_1^2)+p_2^2-2p_1p_3+2q_1q_2-q_2^2}{(q_2\omega_0)^2+(q_3-q_1\omega_0^2)^2}\right]$$

$$\left[\frac{d}{d\tau}\operatorname{Re}(\lambda)\right]=\left[\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\omega_0}=\left[\frac{3\omega_0^4+\omega_0^2(2p_1^2-4p_2-2q_1^2)+p_2^2-2p_1p_3+2q_1q_3-q_2^2}{(q_2\omega_0)^2+(q_3-q_1\omega_0^2)^2}\right]$$

$$\left[\frac{d}{d\tau}\operatorname{Re}(\lambda)\right]>0$$

By using this condition $N_1>0$, $N_2>0$, $N_3>0$ we have $\left[\frac{d}{d\tau}(\operatorname{Re}(\lambda))\right]_{\lambda=i\omega_0}>0$

Therefore, the Hopf bifurcation occurs at $\tau>\tau_0$

6. Numerical Simulation

We study the Hopf bifurcations for the system (2.1) with the tolerance parameter (τ). For the system of equations, the parameters are identified as shown in the example 1. For different values of τ the graphs are shown below.

Example: 6.1 let us choose the following parameters for examination

$a_1=1, a_2=1, a_3=1, \alpha_{12}=0.2, \alpha_{21}=0.5, \alpha_{32}=0.4, k_1=50, k_2=50, k_3=50, x=3, y=3, z=3.$

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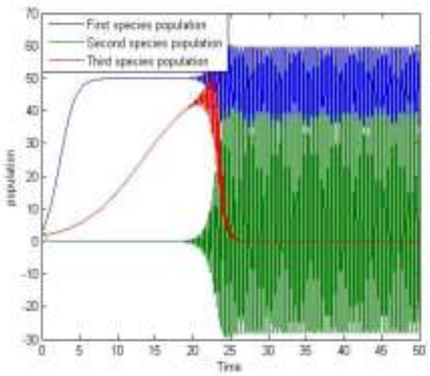


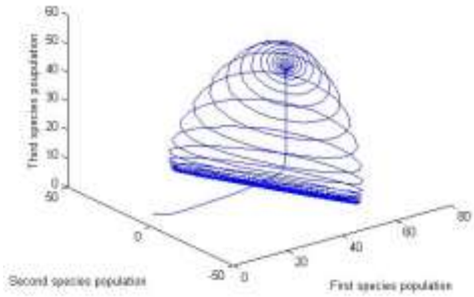
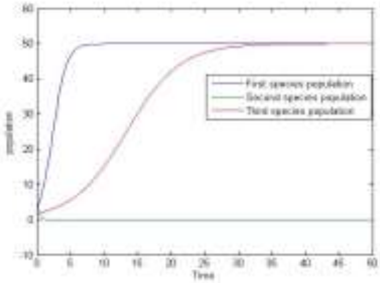


Fig. 6.1 (A) **Fig. 6.1 (B)**

The unbounded periodic solutions for the system (3.B.1.1) when τ = 0.069



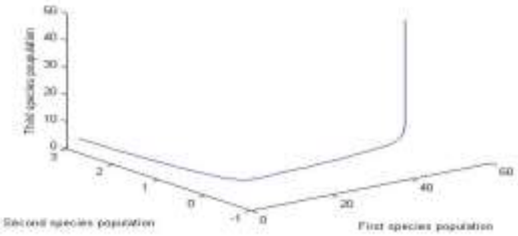
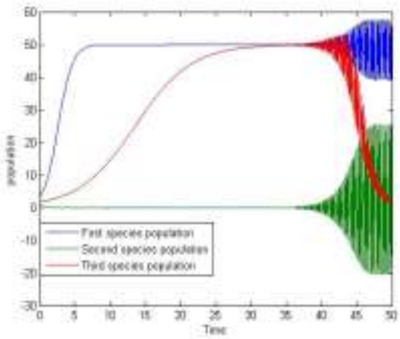


Fig. 6.1 (C) **Fig. 6.1 (D)**

The bounded solutions for the system (2.1) when τ = 0.068

Example: 6.2 let us choose the following parameters for examination

$a_1 = 1, a_2 = 1, a_3 = 1, \alpha_{12} = 0.1, \alpha_{21} = 0.5, \alpha_{32} = 0.4, k_1 = 50, k_2 = 50, k_3 = 50\,, x = 3, y = 1, z = 2.$



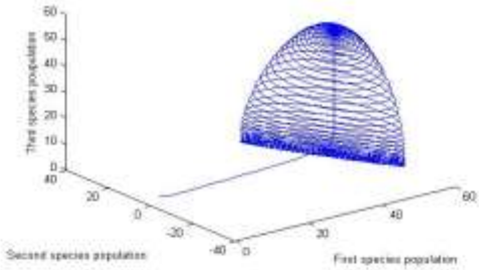
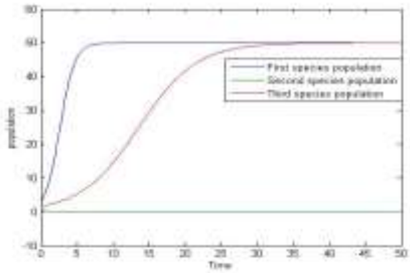


Fig. 6.2 (A) **Fig. 6.2 (B)**

The unbounded periodic solutions for the system (2.1) when τ = 0.065



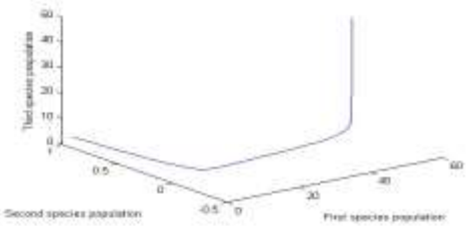


Fig. 6.2 (C) **Fig. 6.2 (D)**

The bounded solutions for the system (2.1) when τ = 0.064

Example : 6.3 Let us choose the following parameters for examination

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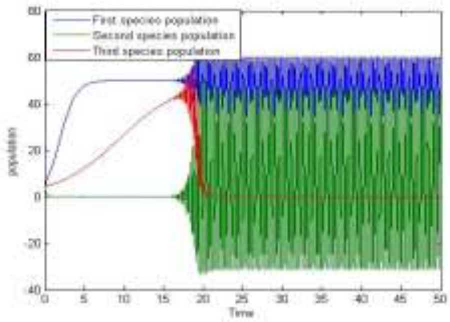
$a_1 = 1, a_2 = 0.5, a_3 = 0.25, \alpha_{12} = 0.1, \alpha_{21} = 0.5, \alpha_{32} = 0.4, k_1 = 50, k_2 = 50, k_3 = 50$, x=5,y=3,z=6.



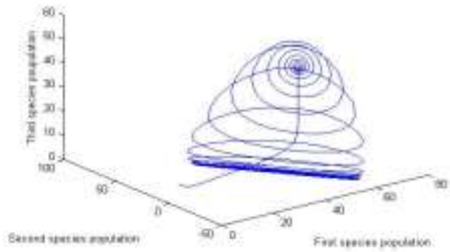
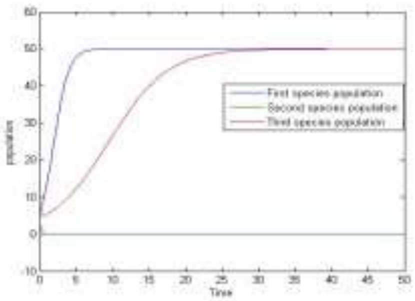


Fig. 6.3 (A) **Fig. 6.3 (B)**

The unbounded periodic solutions for the system (2.1) when τ = 0.072



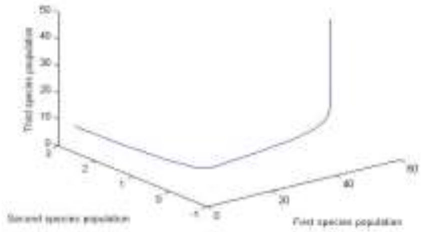
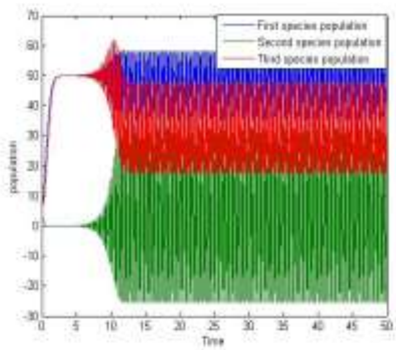


Fig. 6.3 (C) **Fig. 6.3 (D)**

The bounded solutions for the system (2.1) when τ = 0.071

Example: 6.4 Let us choose the following parameters for examination $a_1 = 3, a_2 = 2;, a_3 = 3, \alpha_{12} = 0.2, \alpha_{21} = 0.5, \alpha_{32} = 0.4, k_1 = 50, k_2 = 50, k_3 = 50,$

$$x = 5, y = 3, z = 6.$$



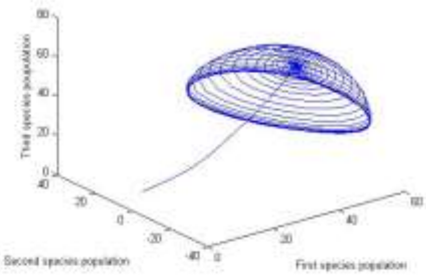
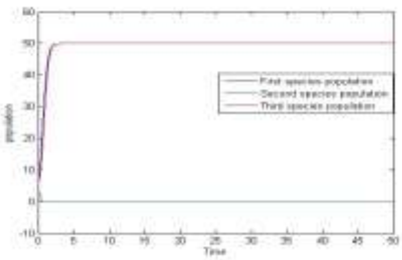


Fig. 6.4 (A) **Fig. 6.4 (B)**

The unbounded periodic solutions for the system (2.1) when τ = 0.65



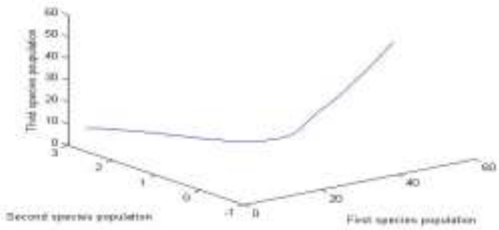


Fig. 6.4 (C) **Fig. 6.4 (D)**

The bounded solutions for the system (2.1) when τ = 0.6

7.Conclusion

A logistics growth model with three species is considered for investigation. The model consists of first (x), second (y) and third (z) species. The delay parameter is incorporated in the interaction of first (x) and second (y) species. The system is locally asymptotically stable at co-existing state. Numerical simulation is carried out for different values of τ and the dynamics was shown with suitable examples. In this model we study the parametric (τ) based Hopf bifurcation. We take four sets of examples to study the bifurcation nature. In each of the examples for different values of the delay parameter 'τ' the dynamics was shown and observes the following. The Hopf bifurcation exists for three examples shown in the following table.

S. No	Example	Hopf bifurcation value
1	Example 6.1	$\tau > 0.068$
2	Example 6.2	$\tau > 0.064$
3	Example 6.3	$\tau > 0.071$
4	Example 6.4	$\tau > 0.6$

Hence the delay parameter τ stabilizes the system.

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