Rogers-Ramanujan-Slater Type Identities

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Abstract: In this paper, we have derived some identities of the Rogers-Ramanujan-Slater's Type by using some Series Transformations most of which are available in Ramanujan's lost notebooks. These transformations can also be derived as limiting cases of transformation between basic hyper geometric series and are also available in the current work of D. Bowman, J. McLaughlin and A. V. Sills. We derive such identities related to modulo 4, 5, 6, 8, 10, 12, 20 and 30 from these transformations with incorporation of some results from the famous list of 130 identities of Lucy Slater and few Entries from Ramanujan's Lost notebook.

Keywords: Basic hyper geometric Series, Transformations, q −series, Rogers-Ramanujan type identity, L. J. Slater, Ramanujan's theta function.

Introduction: The following two Identities, namely for |q|<1

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n},n\neq 0,\pm 2\,(mod\ 5)=\frac{(q^2,q^3,q^5;q^5)_\infty}{(q;q)_\infty}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n},n\neq 0,\pm 1(mod\ 5)=\frac{(q,q^4,q^5;q^5)_\infty}{(q;q)_\infty}$$

where$(q;q)_n=\prod_{j=1}^{n}(1-q^j)$, $(q;q)_\infty=\prod_{j=1}^{\infty}(1-q^j)$,

are the most famous "Series =Product" identities called Rogers-Ramanujan Identity which have motivated extensive research over the past hundred years. There are numerous identities in the literature that are similar to Rogers-Ramanujan Type Identities. Many eminent mathematicians like Rogers, G. E. Andrews and B. C. Berndt [4], W. N. Bailey [10], D. Bowman, J. McLaughlin and A.V. Sills [1] have contributed a lot in this field. Of special note, the work of Lucy Slater has also to be mentioned. The 1952 paper [8] of Lucy Slater contains a list of 130 such famous identities many of them are new. The present study basically attempts to derive some more identities of Rogers-Ramanujan-Slater Type by correlating some identities from Lucy Slater famous list [8] analytically with some transformations between basic hyper geometric series.

Definitions and notations:

For |q|<1, the q-shifted factorial is defined by

$$(a;q)_0=1$$

$(a;q)_n=\prod_{k=0}^{n-1}(1-aq^k)$, for n≥1

and $(a;q)_\infty=\prod_{k=1}^{\infty}(1-aq^k)$.

It follows that $(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$

The multiple q-shifted factorial is defined by

$$(a_1,a_2,....,a_m;q)_n = (a_1;q)_n (a_2;q)_n ....(a_m;q)_n$$

$$(a_1,a_2,....,a_m;q)_\infty = (a_1;q)_\infty (a_2;q)_\infty ....(a_m;q)_\infty .$$

The Basic Hyper geometric Series is

$$_{p+1}\phi_{p+r}\left(\begin{matrix} a_1,a_2,.......,a_{p+1};q;x \\ b_1,b_2,.......,b_{p+r} \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n ....(a_{p+1};q)_n x^n (-1)^{nr} q^{\frac{n(n-1)r}{2}}}{(q;q)_n (b_1:q)_n (b_2;q)_n .....(b_{p+r};q)_n} \quad (1.1)$$

The series $_{p+1}\phi_{p+r}$ converges for all positive integers r and for all x. For r=0 it converges only when |x|<1.

Ramanujan's Theta function: Ramanujan's Theta function ([4], P.11, Eq. (1.1.5)) is defined as

f (a, b) = $\sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}$, for |ab|<1. (1.2)

The following special cases of f (a, b) arise so often that they were given their own notation by Ramanujan ([4], P.11):

φ(q) = f (q,q)

ψ(q) =f (q, q^3)

f $(-q)$= $f(-q, -q^2)$

Jacobi's triple product identity:(See [7], P.2, Eq. (1.1.7)

For |ab| < 1, f (a, b) =$(-a, -b, ab; ab)_\infty$ (1.3)

An immediate corollary ([7], P-2, Eq. (1.1.8), (1.1.9), (1.1.10)) of this identity are the following:

f $(-q)$ = $(q;q)_\infty$ (1.4)

φ$(-$q) =$\frac{(q;q)_\infty}{(-q;q)_\infty}$ (1.5)

$\psi(-$q) = $\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}$ (1.6)

1. **We proceed through some q −series identities and some general series transformations that can be derived as the limiting case of transformations between basic hypergeometric series in this section:**

$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} z^n}{(1-q)(1-q^2)(1-q^n)} = \prod_{m=0}^{\infty}(1 + zq^m)$ (2.1)

(Andrews [3]; p-19, Eq. (2.2.6)

$$\sum_{n=0}^{\infty}\frac{z^n}{(1-q)(1-q^2)(1-q^n)}=\prod_{m=0}^{\infty}\frac{1}{(1-zq^m)} \tag{2.2}$$

(Andrews [3]; p-20, Eq. (2.2.8)

$$\sum_{n=0}^{\infty}\frac{(a,b;q)_n q^{n(n-1)/2}\,(-c\gamma/ab)^n}{(c,\gamma,q;q)_n}=\frac{(c\gamma/ab\,;q)_\infty}{(\gamma;q)_\infty}\sum_{n=0}^{\infty}\frac{(c/a,c/b;q)_n q^{n(n-1)/2}(-\gamma)^n}{(c,c\gamma/ab,q;q)_n} \tag{2.3}$$

$$\sum_{n=0}^{\infty}\frac{(a;q)_n q^{n(n-1)/2}\,\gamma^n}{(b;q)_n(q;q)_n}=\frac{(-\gamma;q)_\infty}{(b;q)_\infty}\sum_{n=0}^{\infty}\frac{(-a\gamma/b;q)_n q^{n(n-1)/2}\,(-b)^n}{(-\gamma;q)_n(q;q)_n} \tag{2.4}$$

$$\sum_{n=0}^{\infty}\frac{(a;q)_n q^{n(n-1)/2}\,\gamma^n}{(q;q)_n}=(-\gamma;q)_\infty\sum_{n=0}^{\infty}\frac{(-a\gamma)^n q^{n(n-1)}}{(-\gamma;q)_n(q;q)_n} \tag{2.5}$$

$$\sum_{n=0}^{\infty}\frac{q^{3n(n-1)/2}\,\gamma^n}{(\gamma;q^2)_n(q;q)_n}=\frac{1}{(\gamma;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{2n^2-n}\gamma^n}{(q^2;q^2)_n} \tag{2.6}$$

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)}(-\gamma)^n}{(\gamma q;q^2)_n(q^2;q^2)_n}=\frac{1}{(\gamma q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2-n}(-\gamma)^n}{(q;q)_n} \tag{2.7}$$

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}(-\gamma)^n}{(b;q)_n(q;q)_n}=(\gamma;q)_\infty\sum_{n=0}^{\infty}\frac{q^{(3n^2-3n)/2}(-b\gamma)^n}{(q;q)_n(b;q)_n(\gamma;q)_n} \tag{2.8}$$

Proof of Identity (2.1):

Let $f(z)=\prod_{m=0}^{\infty}(1+zq^m)$

Then $f(z)=(1+z)\prod_{m=1}^{\infty}(1+zq^m)$=$(1+z)\prod_{m=0}^{\infty}(1+zq^{m+1})$

=$(1+z)\prod_{m=0}^{\infty}\{1+(zq)q^m\}$=$(1+z)f(zq)$

Thus, $f(z)=\prod_{m=0}^{\infty}(1+zq^m)$=$(1+z)f(zq)$ (2.9)

Now in expanded form if we write$f(z)=\sum_{n=0}^{\infty}A_n(q)z^n$, where the coefficients depends on q, then in view of the result (2.9), we have

$$f(z)=(1+z)\sum_{n=0}^{\infty}A_n(q)q^n z^n$$

= $\sum_{n=0}^{\infty}A_n(q)q^n z^n$+$\sum_{n=0}^{\infty}A_n(q)q^n z^{n+1}$

= $\sum_{n=0}^{\infty}A_n(q)q^n z^n$+$\sum_{n=1}^{\infty}A_{n-1}(q)q^{n-1}z^n$

Hence we obtain the following recurrence relation:

$A_n(q)=q^nA_n(q)+q^{n-1}A_{n-1}(q)$, $A_0(q)=1$

$\Rightarrow A_n(q)$=$\frac{q^{n-1}}{1-q^n}A_{n-1}(q)$=$\frac{q^{n-1}.q^{n-2}}{(1-q^n)(1-q^{n-1})}A_{n-2}(q)$, $n\geq 2$

Thus, $A_n(q)$=$\frac{q^{n(n-1)/2}}{(1-q)(1-q^2)(1-q^n)}$

Therefore, $\prod_{m=0}^{\infty}(1+zq^m)=\sum_{n=0}^{\infty}A_n(q)z^n=\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}z^n}{(1-q)(1-q^2)(1-q^n)}$

Proof of Identity (2.2):

Let us take $g(z) = \prod_{m=0}^{\infty} \frac{1}{(1-zq^m)} = \frac{1}{1-z} \prod_{m=1}^{\infty} \frac{1}{(1-zq^m)}$

$= \frac{1}{1-z} \prod_{m=0}^{\infty} \frac{1}{1-(zq)q^m} = \frac{1}{1-z} . g(zq)$

Hence $(1-z). g(z) = g(zq)$ (2.10)

Now in terms of series, let us consider

$f(z) = \sum_{n=0}^{\infty} B_n(q) z^n$ where the coefficients depends on q. Then in view of (2.10) it can be written as

$$(1-z). \sum_{n=0}^{\infty} B_n(q) z^n = \sum_{n=0}^{\infty} B_n(q) q^n z^n$$

which can easily be simplified to

$$\sum_{n=0}^{\infty} B_n(q) z^n - \sum_{n=1}^{\infty} B_{n-1}(q) z^n = \sum_{n=0}^{\infty} B_n(q) q^n z^n$$

It yields the following recurrence:

$$q^n B_n(q) = B_n(q) - B_{n-1}(q), \qquad B_0(q) = 0$$

Hence $B_n(q) = \frac{1}{1-q^n} . B_{n-1}(q), \quad n \geq 1$

$= \frac{1}{(1-q)(1-q^2)(1-q^n)}$

It follows that

$$\prod_{m=0}^{\infty} \frac{1}{(1-zq^m)} = g(z) = \sum_{n=0}^{\infty} B_n(q) z^n = \sum_{n=0}^{\infty} \frac{z^n}{(1-q)(1-q^2)(1-q^n)}$$

Proof of identities ((2.3) to (2.8)): The transformation (2.3) is a limiting case of q −analogue of Kummer-Thomae-Whipple formula (see [2], page 72, equation 3.2.7).The transformation (2.4) is found in Ramanujan's lost Notebook [9] and a proof can be found in the recent book by Andrews and Berndt [4]. Equation (2.5) follows from (2.4) upon letting $b \to 0$.The proof of (2.6), (2.7) and (2.8) can be found in [6] and an alternative proof can be found in [1].

The transformations (2.1) and (2.2) yield the identities (2.1.1) to (2.1.4) upon some simple substitutions of the parameters:

For instance, the identity (2.1) for $z = q$ yields

$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n} = \prod_{m=0}^{\infty} (1 + q^{m+1}) = (-q;q)_\infty$

$= \frac{(q;q)_\infty}{(q;q)_\infty/(-q;q)_\infty} = \frac{f(-q)}{\varphi(-q)}$ (2.1.1)

which already appears in Slater's list ([8], p.152, Eq. (2))

Replacing q by $q^2$and then setting $z = -q$ in (2.1) we get,

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(q^2;q^2)_n}=\prod_{m=0}^{\infty}(1-q^{2m+1})$$

$=(q;q^2)_\infty$ (Appeared in Slater's list ([8], p.152, Eq. (3))

$$=\frac{(q;q^2)_\infty\,(q;q^2)_\infty\,(q^2;q^2)_\infty}{(q;q)_\infty}$$

$$=\frac{(q,q,q^2;q^2)_\infty}{(q;q)_\infty} \qquad (2.1.2)$$

Replacing q by $q^2$and then setting $z=q$ in (2.1) we get,

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^2;q^2)_n}=\prod_{m=0}^{\infty}(1+q^{2m+1})=(-q;q^2)_\infty$$

$$=\frac{(-q;q^2)_\infty\,(q;q^2)_\infty}{(q;q^2)_\infty}=\frac{(q^2;q^4)_\infty}{(q;q^2)_\infty}$$

$$=\frac{(q^2;q^4)_\infty(q^2;q^4)_\infty(q^4;q^4)_\infty}{(q;q^2)_\infty(q^2;q^4)_\infty(q^4;q^4)_\infty}$$

$$=\frac{(q^2,q^2,q^4;q^4)_\infty}{(q;q)_\infty} \qquad (2.1.3)$$

Finally setting q by $q^2$and $z=q^{1/2}$ in (2.1) we get

$$\sum_{n=0}^{\infty}\frac{q^{(2n^2-n)/2}}{(q^2;q^2)_n}=\prod_{m=0}^{\infty}(1+Q^{4m+1}),where\ \ Q=q^{1/2}$$

$$=(1+Q)(1+Q^5)(1+Q^9)(1+Q^{13})\ldots\ldots\ldots$$

$$=\frac{(Q^2;Q^8)_\infty}{(Q;Q^4)_\infty}=\frac{(Q^2;Q^8)_\infty(Q^6;Q^8)_\infty(Q^8;Q^8)_\infty}{(Q;Q^4)_\infty(Q^6;Q^8)_\infty(Q^8;Q^8)_\infty}$$

Now replacing $Q=q^{1/2}$ and then after some simplification, we obtain the following identity:

$$\frac{(q^{1/2};q^2)_\infty}{(q,q^2;q^4)_\infty}\sum_{n=0}^{\infty}\frac{q^{(2n^2-n)/2}}{(q^2;q^2)_n}=\frac{(q,q^3,q^4;q^4)_\infty}{(q;q)_\infty} \qquad (2.1.4)$$

2. **Identities of Rogers-Ramanujan-Slater Type**: In this section we have derived some identities of the Rogers-Ramanujan-Slater Type by using the transformations (2.4) to (2.8) with various substitutions of the parameters$a,b,c,\gamma,q\in\mathbb{C}$, ($|q|<1$), as involved and by incorporating few identities from Lucy Slater's famous list [8] of 130 identities of Rogers-Ramanujan Type and few entries given by Ramanujan in his lost notebook.
3. **Identities related to modulo 4:**

Setting $a=-q,\ \gamma=q$ in (2.5), we get

$$\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{n(n+1)/2}}{(q;q)_n}=(-q;q)_\infty.\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q^2;q^2)_n} \qquad (3.1.1)$$

On using the identity ([8], p.153, Eq. (7)) and ([8], p.153, Eq. (8)) respectively from Slater list in (3.1.1), we can obtain the following two identities related to modulo 4, which already exists in the literature:

$$\frac{1}{(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{n(n+1)/2}}{(q;q)_n}=\frac{(q,q^3,q^4;q^4)_\infty}{(q;q)_\infty} \qquad (3.1.2)$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n} = \frac{(q,q^3,q^4;q^4)_\infty}{(q;q)_\infty} \tag{3.1.3}$$

Taking $q \to q^2$ and then setting $a = q^2,\ \gamma = -q$ in (2.5), we get after some simplification, the following equation:

$$\frac{1}{(q;q^2)_\infty}\left[\sum_{n=0}^{\infty}(-1)^n q^{n^2} + (q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n^2+3n+1}}{(q;q)_{2n+1}}\right] = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q;q)_{2n+1}} \tag{3.1.4}$$

Now upon using the identity ([8], p.153, Eq. (9)) from Slater list in (3.1.4), we get:

$$\frac{(-q;q)_\infty}{(q;q^2)_\infty}\left[\sum_{n=0}^{\infty}(-1)^n q^{n^2} + (q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n^2+3n+1}}{(q;q)_{2n+1}}\right] = \frac{(-q,-q^3,q^4;q^4)_\infty}{(q;q)_\infty} \tag{3.1.5}$$

Identities related to modulo 6:

Taking $q \to q^2$ and then replacing $a = -q,\ b = -q^2$ and $\gamma = q$ in the transformation (2.4), we get

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty} \cdot \sum_{n=0}^{\infty} \frac{(-1;q^2)_n q^{n^2+n}}{(-q;q^2)_n (q^2;q^2)_n} \tag{3.1.6}$$

Now comparing the equation (3.1.6) with the identity as appeared in Slater list ([8], p.154, Eq. (25)) viz,

$$\prod (1-q^{6n-3})^2 (1-q^{6n}) = \prod \frac{(1-q^{2n})}{(1-q^{2n-1})} \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^4;q^4)_n} \tag{3.1.7}$$

we can obtain the following identity, after some simplification:

$$(-q;q^2)_\infty \cdot \sum_{n=0}^{\infty} \frac{(-1;q^2)_n q^{n^2+n}}{(-q;q^2)_n (q^2;q^2)_n} = \frac{(q^3,q^3,q^6;q^6)_\infty}{(q;q)_\infty} \tag{3.1.8}$$

Taking $q \to q^2$ and then replacing $a = -q,\ b = q$ and $\gamma = q$ in the transformation (2.4), we get

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n\, q^{n^2}}{(q;q)_{2n}} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n\, (q;q^2)_n\, q^{n^2}}{(-q;q^2)_n (q^2;q^2)_n} \tag{3.1.9}$$

Comparing (3.1.9) with the identity ([8], p.155, Eq. (29)) from Slater List, the following identity can be obtained:

$$(q^2;q^2)_\infty \cdot \sum_{n=0}^{\infty} \frac{(-1)^n\, (q;q^2)_n\, q^{n^2}}{(-q;q^2)_n (q^2;q^2)_n} = \frac{(-q^2,-q^4,q^6;q^6)_\infty}{(-q;q)_\infty} \tag{3.1.10}$$

Again, Taking $q \to q^2$ and then replacing $a = -q^2,\ b = q^3$ and $\gamma = q^2$ in the transformation (2.4), we get

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n(n+1)}}{(q^3;q^2)_n (q^2;q^2)_n} = \frac{(-q^2;q^2)_\infty}{(q^3;q^2)_\infty} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2+2n}}{(-q^2;q^2)_n (q^2;q^2)_n}$$

which, after some simplification, reduces to

$$\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2+2n}}{(-q^2;q^2)_n (q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n(n+1)}}{(q;q)_{2n+1}} \tag{3.1.11}$$

On comparing the equation (3.1.11) with the identity as appeared in (Ramanujan [10, Ent. 4.2.13], p. 88), We get,

$$\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}.\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n q^{n^2+2n}}{(q^4;q^4)_n} = \frac{f(q,q^5)}{\varphi(-q^2)} \tag{3.1.12}$$

Hence, finally, we reach to the identity (3.1.13) after using (1.4) and (1.5):

$$\frac{(-q;q)_\infty}{(q;q^2)_\infty}.\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n q^{n^2+2n}}{(q^4;q^4)_n} = \frac{(-q,-q^5,q^6;q^6)_\infty}{(q;q)_\infty} \tag{3.1.13}$$

Identities related to modulo 8: From the transformation (2.5), the following identities related to modulo 8 have been derived as follows:

For $q \to q^2, a = -q,\ \gamma = q^3$ in (2.5), it gives:

$$(-q;q)_\infty \sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(-q;q^2)_{n+1}(q^2;q^2)_n} = \frac{(q,q^7,q^8;q^8)_\infty}{(q;q)_\infty} \tag{3.1.14}$$

(Using ([8], p.155, Eq. (34)) from Slater list)

For $q \to q^2, a = -q,\ \gamma = q$ in (2.5), it gives:

$$(-q;q)_\infty \sum_{n=0}^{\infty}\frac{q^{2n^2}}{(-q;q^2)_n(q^2;q^2)_n} = \frac{(q^3,q^5,q^8;q^8)_\infty}{(q;q)_\infty} \tag{3.1.15}$$

(Using ([8], p.155, Eq. (36)) from Slater list)

For $q \to q^2, a = q,\ \gamma = -q^3$ in (2.5), it gives:

$$\frac{(-q;q)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n} = \frac{(q,q^7,q^8;q^8)_\infty}{(q;q)_\infty} \tag{3.1.16}$$

(Using ([8], p.155, Eq. (38)) from Slater list)

For $q \to q^2, a = q,\ \gamma = -q$ in (2.5), it gives:

$$\frac{1}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty} \tag{3.1.17}$$

(Using ([8], p.156, Eq. (39)) from Slater list)

Identities related to modulo 10:

Setting $\gamma = q^3$ in (2.6) and incorporating ([8], p.156, Eq. (44)) from the Slater list, we get the following identity on some simplification:

$$\frac{1}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(q^2;q^2)_n} = \sum_{n=0}^{\infty}\frac{q^{(3n^2+3n)/2}}{(q;q^2)_{n+1}(q;q)_n} = \frac{(q^2,q^8,q^{10};q^{10})_\infty}{(q;q)_\infty} \tag{3.1.18}$$

Similarly, for $\gamma = q$ in (2.6) and using ([8], p.156, Eq. (46)) from the Slater list, we get

$$\frac{1}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q^2;q^2)_n} = \frac{(q^4,q^6,q^{10};q^{10})_\infty}{(q;q)_\infty} \tag{3.1.19}$$

Taking $q \to q^2$ and then setting $b = q^3,\ \gamma = -q^3$ in (2.8), we get

$$\sum_{n=0}^{\infty}\frac{(1-q)q^{n^2+2n}}{(q;q^2)_{n+1}(q^2;q^2)_n} = (-q^3;q^2)_\infty \sum_{n=0}^{\infty}\frac{q^{3n(n+1)}}{(q^2;q^2)_n\,(q^6;q^4)_n} \tag{3.1.20}$$

Now, taking $q \to q^{1/2}$ in (3.1.20) and then using the identity ([8], p.156, Eq. (44)), we get the following identity on some simplification:

$$\frac{1}{(1+q^{1/2})(-q^{3/2};q)_\infty}\sum_{n=0}^{\infty}\frac{q^{(n^2+2n)/2}}{(q^{1/2};q)_{n+1}(q;q)_n}=\frac{(q^2,q^8,q^{10};q^{10})_\infty}{(q;q)_\infty} \quad (3.1.21)$$

Identities related to modulo 12:

For, $q\to q^2$, $a=-1,\ b=q$ and $\gamma=q^2$ in (2.4), we get

$$\sum_{n=0}^{\infty}\frac{(-1;q^2)_n q^{n(n+1)}}{(q;q)_{2n}}=\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}.\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n\,.q^{n^2}}{(q^4;q^4)_n} \quad (3.1.22)$$

Using the Identity from Slater list ([8], p.156, Eq. (48)) in (3.1.22), we get the following identity related to modulo 12:

$$\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}.\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n\,.q^{n^2}}{(q^4;q^4)_n}=\frac{(q^5,q^7,q^{12};q^{12})_\infty}{(q;q)_\infty}-q\,\frac{(q^1,q^{11},q^{12};q^{12})_\infty}{(q;q)_\infty} \quad (3.1.23)$$

Identities related to modulo 20:

Taking $q\to q^2$ and $b=-q,\ \gamma=q$ in (2.8), and then using ([8], p.156, Eq. (46) after replacing q by q^2), the following identity can be obtained:

$$\frac{(-q;q)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(-q;q^2)_n(q^2;q^2)_n}=\frac{(q^8,q^{12},q^{20};q^{20})_\infty}{(q;q)_\infty} \quad (3.1.24)$$

For $q\to q^2$ and $b=q,\ \gamma=-q$ in (2.8), and using ([8], p.160, Eq. (79)) from Slater list, we get:

$$(-q:q)_\infty\sum_{n=0}^{\infty}\frac{q^{3n^2-n}}{(q^2;q^2)_n(q^2;q^4)_n}=\frac{(q^8,q^{12},q^{20};q^{20})_\infty}{(q;q)_\infty} \quad (3.1.25)$$

For $q\to q^2$ and $b=q^3,\ \gamma=-q^2$ in (2.8), the following identity can be obtained: $(-q^2:q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{3n^2+2n}}{(q^4;q^4)_n(q;q^2)_{n+1}}=\frac{(q^3,q^7,q^{10};q^{10})_\infty(q^4,q^{16};q^{20})_\infty}{(q;q)_\infty}$ (3.1.26)

(On using ([8], p.162, Eq. (94)) from Slater list)

Finally upon letting $\to q^2$ and $b=q,\ \gamma=-q^2$ in (2.8), and then using ([8], p.162, Eq. (99)) from Slater list, we get the following:

$$(-q^2:q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{3n^2}}{(q^4;q^4)_n(q;q^2)_n}=\frac{(q^1,q^9,q^{10};q^{10})_\infty(q^8,q^{12};q^{20})_\infty}{(q;q)_\infty} \quad (3.1.27)$$

Identities related to modulo 30:

For $a=0,\ b=q^{1/2}$ and $\gamma=q$ in (2.4), we get

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^{1/2};q)_n(q;q)_n}=\frac{(-q;q)_\infty}{(q^{1/2};q)_\infty}.\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2/2}}{(q^2;q^2)_n} \quad (3.1.28)$$

Now, taking $q\to q^2$ in (3.1.28) and then using ([8], p.162, Eq. (99)) from Slater list, we get the identity related to modulo 30 as follows:

$$\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}.\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(q^4;q^4)_n}=\frac{(-q^{13},-q^{17},q^{30};q^{30})_\infty}{(q;q)_\infty}-q\,\frac{(-q^7,-q^{23},q^{30};q^{30})_\infty}{(q;q)_\infty} \quad (3.1.29)$$

For $q\to q^2,\ a=0, b=q^3,\ \gamma=q^2$ in (2.4), we get the following identity after:

$$\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}.\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+2n}}{(q^4;q^4)_n}=\frac{(-q^{11},-q^{19},q^{30};q^{30})_\infty}{(q;q)_\infty}-q^3\,\frac{(-q^1,-q^{29},q^{30};q^{30})_\infty}{(q;q)_\infty} \quad (3.1.30)$$

(On using ([8], p.162, Eq. (94)) from Slater list)

For $q \to q^2$ and $b = -q^2,\ \gamma = q^3$ in (2.8) and simplifying a few steps, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(q^4;q^4)_n} = (q;q^2)_\infty \sum_{n=0}^{\infty} \frac{(-q;q^2)_{n+1} q^{3n^2+2n}}{(q^4;q^4)_n (q^2;q^4)_{n+1}} = (q;q^2)_\infty \sum_{n=0}^{\infty} \frac{(-q;q^2)_{n+1} q^{3n^2+2n}}{(q^2;q^2)_{2n+1}} \quad (3.1.31)$$

Now, upon using the identity ([8], p.162, Eq. (97)) from Slater list in (3.1.31), we obtain:

$$(q^2:q^4)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(q^4;q^4)_n} = \frac{(-q^5,-q^{25},q^{30};q^{30})_\infty - q^3(-q^3,-q^{27}q^{30};q^{30})_\infty}{(q^2;q^2)_\infty} \quad (3.1.32)$$

Finally for $q \to q^2$ and $b = q,\ \gamma = -q$ in (2.8), the following identity can be obtained on using ([8], p.162, Eq. (98)) from Slater list:

$$(-q:q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{3n^2-n}}{(q^2;q^2)_n (q^2;q^4)_n} = \frac{(-q^{14},-q^{16},q^{30};q^{30})_\infty - q^2(-q^4,-q^{26};q^{30})_\infty}{(q;q)_\infty} \quad (3.1.33)$$

Identities related to modulo 5:

For $q \to q^2$and $a = 0,\ b = -q, \gamma = q^2,$ (2.4) yields

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_n (q^2;q^2)_n} = \frac{(-q^2;q^2)_\infty}{(-q;q^2)_\infty} \cdot \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n} \quad (3.1.34)$$

which imply, upon using ([8], p.154, Eq. (20)), the following identity:

$$(-q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_n (q^2;q^2)_n} = \frac{(q^2,q^3,q^5;q^5)_\infty}{(q;q)_\infty} \quad (3.1.35)$$

The transformation (2.6) for $q \to q^{1/2}$ and $\gamma = q^{3/2}$ yield the following identity on some simplification and by using ([8], p.153, Eq. (14)):

$$(q^{3/2};q)_\infty \sum_{n=0}^{\infty} \frac{q^{3n(n+1)/4}}{(q^{3/2};q)_n (q^{1/2};q^{1/2})_n} = \frac{(q^1,q^4,q^5;q^5)_\infty}{(q;q)_\infty} \quad (3.1.36)$$

Setting $\gamma = q$ and then taking $q \to q^{1/2}$ in (2.6) we get the following identity with the incorporation of the Slaters identity ([8], p.154, Eq. (18)):

$$(q^{1/2};q)_\infty \sum_{n=0}^{\infty} \frac{q^{(3n^2-n)/4}}{(q^{1/2};q)_n (q^{1/2};q^{1/2})_n} = \frac{(q^2,q^3,q^5;q^5)_\infty}{(q;q)_\infty} \quad (3.1.37)$$

Conclusion: The methodology used in the present paper may be used to derive such more identities by correlating some more transformations between basic hyper geometric series of this type or may be from various entries given by Ramanujan in his lost notebooks. During the study, some dependency phenomenon has been observed among the identities which have not brought under the purview of the present study.

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