ISSN: 2278-4632
Vol-14, Issue-10, No.03, October: 2024

SOME RESULTS ON RADIO HARMONICAL MEAN GRAPHS

Nithin J V, PET Research Scholar, Mandya-571401 Email: nithinjvpatel@gmail.com,
Shanmukha B , Department of Mathematics, PESCE, Mandya -571401
Email:shanmukhmaths@gmail.com

ABSTRACT:

A radio Harmonic Mean labeling of a connected graph G is a one-to-one map f from the vertex set V (G) to the set of natural numbers N such that for two distinct vertices u and v of G,

$$d(u, v)+ \left\lceil\frac{2f(u)f(v)}{f(u)+f(v)}\right\rceil \geq 1+\text{diam } G .$$

The radio Harmonic Mean number of f, rhmn(f) is the maximum number assigned to any vertex of G. The radio Harmonic Mean number of G, rhmn(G) is the minimum value of rhmn(f) taken over all radio Harmonic Mean labeling f of G. In this paper we have determined the radio Harmonic Mean number of some graphs are discussed.

INTRODUCTION :

The mean defined as a midway between the value of other quantities or as an average which is important for the researchers of Mathematics and Statistics for their investigations and justifications. The Arithmetic Mean, Geometric Mean and Harmonic Mean are the three classical means among ten Greek means which are defined on the basis of proportions. These means were studied by Pythagoreans and later developed by Greek Mathematicians. because of their importance in geometry and music.

Mean: A mean is defined as a function M: $R_+^2 \to R_+$ which has the property

$$\min(x_1, x_2) \leq M(x_1, x_2) \leq \max(x_1, x_2)$$

where x_1 and x_2 are positive real numbers.

Recently, the researchers have effectively utilized Mathematical means for labeling the graphs, for processing the digital images. According to Wang, Bin Yao and Ming Yao, graph labelings are used for incorporating redundancy in disks, designing drilling machines, creating layouts for circuit boards, and configuring resistor networks. Labeled graphs serve as useful models for a broad range of applications such as coding theory, X-ray crystallography, astronomy, circuit design, communication network addressing, database management, secret sharing schemes and models for constraint programming over finite domains.

In [9], the notion of mean labeling was introduced by Somasundaram and Poonraj and also studied the concept of a Geometric Mean labeling of a graph G. with p vertices and q edges and proved paths, cycles, combs, ladders are Geometric Mean graphs and $K_n(n > 4)$ and $K_{1,n}(n > 5)$ are not Geometric Mean graphs. Authors proved $C_m \cup P_n$; $C_m \cup C_n$; nK_3; $nK_3 \cup P_n$; $nK_3 \cup C_m$; P_n^2 and crowns are Geometric Mean graphs. Authors investigated Geometric Mean labelings in the context of duplication of graph elements in cycle C_n and path P_n. According to Beineke and Hegde, graph labeling serves as a frontier between number theory and structure of graphs. Authors [4] proved that some disconnected graphs are Harmonic Mean graphs. In [10], Contra Harmonic Mean labeling of a graph is introduced and also investigated that for a polygonal chain, square of the path and dragon are Harmonic Mean graph of order at most 5. Abundant literature exists as of today concerning the structure of graphs admitting a variety of function assigning real numbers to their elements so that given conditions are satisfied. Authors in study of vertex functions f : V (G) → A,A ⊆ N for which the induced edge function f^* : E(G) → N is defined as G,

$$f^*(uv) = \left\lceil\frac{2[f(u)^2+f(u)f(v)+f(v)^2}{3(f(u)+f(v))}\right\rceil \quad \text{or} \quad f^*(uv) = \left\lfloor\frac{2[f(u)^2+f(u)f(v)+f(v)^2}{3(f(u)+f(v))}\right\rfloor$$

for every u,v ∈ E(G) are all distinct known as Centroidal Mean Labeling and introduced Centroidal Mean Labeling of some standard graphs like triangular snake, Double triangular snake, triangular ladder and so on.

Graph: A graph G is a pair (V, E), where V is a nonempty set and E is a set of unordered pairs of elements taken from the set V. A graph which does not contain loops and multiple edges is a simple graph, a finite number of vertices and edges in a graph is a finite graph and undirected with p vertices and q edges. The cardinality of vertex set V of a graph is the order and the cardinality of edge set E is called the size of the graph G. The graph G − e is obtained from G by deleting an edge e. The distance between two vertices x and y of G is denoted by d(x, y) and diam(G) indicate the diameter of G.

The distance d(u,v) from a vertex u to a vertex v in a connected graph G is the minimum of the lengths of the u − v paths in G.

The eccentricity e(v) of a vertex v in a connected graph G is the distance between v and a vertex farthest from v in G.

The diameter diam(G) of graph G is the greatest eccentricity among the vertices of G.

RADIO HARMONIC MEAN GRAPH:

A radio Harmonic Mean labeling of a connected graph G is one to one map f from the vertex set V (G) of G to set of natural numbers N such that for two distinct vertices u and v of G satisfies the condition G, $d(u, v)+ \left\lceil \frac{2f(u)f(v)}{f(u)+f(v)} \right\rceil \geq 1+\text{diam } G$. A graph which admits radio Harmonic Mean labeling is called radio Harmonic Mean graph.

RADIO HARMONIC MEAN NUMBER OF A GRAPH:

A Radio Harmonic Mean number of graph G is denoted by rhmn(G). It is defined as the lowest span taken over all radio labeling of graph G. The span of a labeling f is the maximum integer that f maps to a vertex of graph G.

For other terminology and notations refer [3].

RADIO HARMONIC MEAN LABELING OF A GRAPH :

In this section the Radio Harmonic Mean labeling of graphs containing cycles such as triangular snake $T_n {}^J K_1$, double triangular snake $D_n(T_n)^J K_1$, and $TL_n {}^J K_1$, are discussed using the following definition.

Definition 2.1. [7] A radio Harmonic Mean labeling of a connected graph G is one to one map f from the vertex set V (G) of G to set of natural numbers N such that for two distinct vertices u and v of G satisfies the condition G, $d(u, v)+ \left\lceil \frac{2f(u)f(v)}{f(u)+f(v)} \right\rceil \geq 1+\text{diam } G$. A graph which admits radio Harmonic Mean labeling is called radio Harmonic Mean graph.

MAIN RESULTS :

Theorem 3.1. A radio Harmonic Mean number of the graph Triangular snake $T_n {}^J K_1$ is $5n - 4$ for $n \geq 2$.

Proof. Consider a triangular snake T_n. Let u_i, $v_i, 1 \leq i \leq n$ be the vertices of a triangular snake graph T_n. Join u_iv_i and $u_{i+1}v_i$. Let w_i, x_i be the pendent vertices. Join u_iw_i and v_ix_i , $1 \leq i \leq n$. The resultant graph is $T_n {}^J K_1$, whose edge set is

$E(T_n {}^J K_1) = \{u_iu_{i+1}, u_iv_i, v_iu_{i+1}, v_ix_i | 1 \leq i \leq n - 1\} \cup \{u_iw_i | 1 \leq i \leq n - 1\}$ and diameter of $T_n {}^J K_1$ is $D = n + 1$. The function $f : V (T_n {}^J K_1) \rightarrow$

1,2, 3...(p + q) is defined by

$f(u_i) = 4n+i-4, \quad 1 \leq i \leq n$

$f(v_i) = 3n+i-3, \quad 1 \leq i \leq n-1$

 ISSN: 2278-4632
Vol-14, Issue-10, No.03, October: 2024

$f(w_i) = n+i-2, \quad 1 \le i \le n$

$f(x_i) = 2n+i-2, \quad 1 \le i \le n-1$

Case(a): Consider the pair $(u_i,u_j), i \neq j, 1 \le i,j \le n$

$$d(u_i, v_j)+ \left\lceil \frac{2f(u_i)f(u_j)}{f(u_i)+f(u_j)} \right\rceil \ge 1+(n+1) = (n+2)$$

$$d(u_i, v_j)+ \left\lceil \frac{2(4n+i-\ )(4n+j+4)}{8n+i+j-8} \right\rceil \ge 1+(n+1) = (n+2)$$

Case(b): Consider the pair $(w_i,w_j), i \neq j, 1 \le i,j \le n$

$$d(u_i, v_j)+ \left\lceil \frac{2f(u_i)f(u_j)}{f(u_i)+f(u_j)} \right\rceil \ge 1+(n+1) = (n+2)$$

$$d(u_i, v_j)+ \left\lceil \frac{2(4n+i-4)(4n+j+4)}{8n+i+j-8} \right\rceil \ge 1+(n+1) = (n+2)$$

Case(c): Consider the pair $(v_i,v_j), i \neq j, 1 \le i,j \le n$

$$d(u_i, v_j)+ \left\lceil \frac{2f(v_i)f(v)}{f(v_i)+f(v_j)} \right\rceil \ge 1+(n+1) = (n+2)$$

$$d(u_i, v_j)+ \left\lceil \frac{2(3n+i-3)(3n+j+3)}{6n+i+j-6} \right\rceil \ge 1+(n+1) = (n+2)$$

Case(d): Consider the pair $(x_i,x_j), i \neq j, 1 \le i,j \le n$

$$d(u_i, v_j)+ \left\lceil \frac{2f(x_i)f(x_j)}{f(x_i)+f(x_j)} \right\rceil \ge 1+(n+1) = (n+2)$$

$$d(u_i, v_j)+ \left\lceil \frac{2(2n+i-2)(2n+j+4)}{4n+i+j-4} \right\rceil \ge 1+(n+1) = (n+2)$$

Case(e): Consider the pair $(u_i,v_j), 1 \le i \le n, 1 \le j \le n-1$

$$d(u_i, v_j)+ \left\lceil \frac{2f(u_i)f(v_j)}{f(u_i)+f(v_j)} \right\rceil \ge 1+(n+1) = (n+2)$$

$$d(u_i, v_j)+ \left\lceil \frac{2(4n+i-4)(3n+j+3)}{7n+i+j-7} \right\rceil \ge 1+(n+1) = (n+2)$$

Case(f): Consider the pair $(u_i,w_j), 1 \le i,j \le n$

$$d(u_i, v_j)+ \left\lceil \frac{2f(u_i)f(w_j)}{f(u_i)+f(w_j)} \right\rceil \ge 1+(n+1) = (n+2)$$

$$d(u_i, v_j)+ \left\lceil \frac{2(4n+i-4)(4n+j+4)}{5n+i+j-6} \right\rceil \ge 1+(n+1) = (n+2)$$

The remaining pairs $(w_i,v_j),(w_i,x_j),(x_i,x_j)$ also satisfied the radio Harmonic Mean condition. Thus, the radio Harmonic Mean condition is satisfied for all pairs of vertices. Hence f is a valid radio Harmonic Mean labeling of Triangular snake
$T_n{}^J K_1$. Therefore rhmn $(T_n{}^J K_1) = 5n - 4, n \ge 2$. □

Example 3.1. Consider the triangular snake $T_3{}^J K_1$. The following figure shows the radio Harmonic Mean labeling of a graph with radio Harmonic Mean number 24.

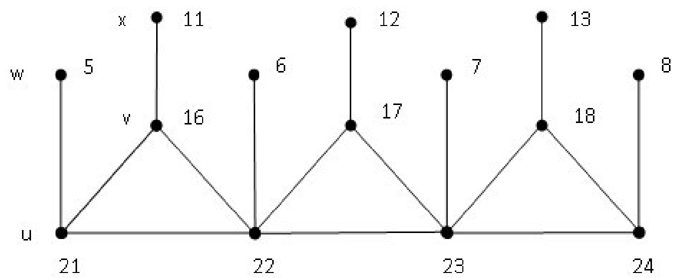

Figure 1. Triangular snake $T_3\, {}^{J}K_1$

Theorem 3.2. The radio Harmonic Mean number of the graph $Q_n\, {}^{J}K_1$ is $7n-6$ for $n \geq 2$.

Proof. Let $u_i, 1 \leq i \leq n$ and $w_i, y_i, 1 \leq i \leq n-1$ be the vertices of the quadrilateral snake graph Q_n. Introduce new vertices v_i, x_i and z_i to $y_i, 1 \leq i \leq n-1$ respectively. The resultant graph is $Q_n\, {}^{J}K_1$ whose edge set $E(Q_n\, {}^{J}K_1) = \{\{u_i,u_{i+1}\},\{u_i,w_i\},\{w_i,y_i\},\{y_i,u_{i+1}\},\{w_i,x_i\},\{y_i,z_i\} | 1 \leq i \leq n-1\} \cup \{\{u_i,v_i\} | 1 \leq i \leq n\}$ and diameter of $Q_n\, {}^{J}K_1$ is $D = n + 2$.

Define a function $f : V(Q_n\, {}^{J}K_1) \rightarrow N$ by

$f(u_i) = 6n+i-6, \quad 1 \leq i \leq n$

$f(v_i) = n+i-2, \quad 1 \leq i \leq n$

$f(w_i) = 5n+i-5, \quad 1 \leq i \leq n-1$

$f(x_i) = 2n+i-2, \quad 1 \leq i \leq n-1$

$f(y_i) = 4n+i-4, \quad 1 \leq i \leq n-1$

$f(z_i) = 3n+i-3, \quad 1 \leq i \leq n-1$

now we verify the radio Harmonic Mean condition for f by the condition

$$d(u, v)+\left\lceil\frac{2f(u)f(v)}{f(u)+f(v)}\right\rceil \geq 1+\text{diam } G .$$

Case(1): Consider the pair $(u_i,u_j), i \neq j; \quad 1 \leq i,j \leq n$

$$d(u_i , u_j) + \left\lceil\frac{2f(u_i)f(u_j)}{f(u_i)+f(u_j)}\right\rceil = 1+\left\lceil\frac{2(6n+i-6)(6n+j-6)}{12n+i+j-12}\right\rceil = 1+(n+2) \geq (n+3)$$

Case(2): Consider the pair $(v_i,v_j), i \neq j; \quad 1 \leq i,j \leq n$

$$d(v_i , v_j) + \left\lceil\frac{2f(v_i)f(v_j)}{f(v_i)+f(v_j)}\right\rceil = 3+\left\lceil\frac{2(n+i-2)(n+j-2)}{2n+i+j-4}\right\rceil = 3+(n+2) \geq (n+3)$$

Case(3): Consider the pair $(w_i,w_j), i \neq j; \quad 1 \leq i,j \leq n$

$$d(w_i , w_j) + \left\lceil\frac{2f(w_i)f(w_j)}{f(w_i)+f(w_j)}\right\rceil = 3+\left\lceil\frac{2(5n+i-5)(5n+j-5)}{10n+i+j-10}\right\rceil = 3+(n+2) \geq (n+3)$$

Case(4): Consider the pair $(x_i,x_j), i \neq j; \quad 1 \leq i,j \leq n$

$$d(x_i , x_j) + \left\lceil\frac{2f(x_i)f(x_j)}{f(x_i)+f(x_j)}\right\rceil = 3+\left\lceil\frac{2(2n+i-2)(2n+j- \ )}{4n+i+j-4}\right\rceil = 5+(n+2) \geq (n+3)$$

Case(5): Consider the pair $(y_i,y_j), i \neq j; \quad 1 \leq i,j \leq n$

$$d(y_i , y_j) + \left\lceil\frac{2f(y_i)f(y_j)}{f(y_i)+f(y_j)}\right\rceil = 3+\left\lceil\frac{2(4n+i-4)(4n+j-4)}{8n+i+j-8}\right\rceil = 3+(n+2) \geq (n+3)$$

Case(6): Consider the pair $(z_i,z_j), i \neq j; \quad 1 \leq i,j \leq n$

$$d(z_i , z_j) + \left\lceil\frac{2f(z_i)f(z_j)}{f(z)+f(z_j)}\right\rceil = 3+\left\lceil\frac{2(3n+i- \ )(3n+j- \ )}{6n+i+j-6}\right\rceil = 5+(n+2) \geq (n+3)$$

ISSN: 2278-4632
Vol-14, Issue-10, No.03, October: 2024

Case(7): Consider the pair (u_i,v_j), $1 \le i,j \le n$

$$d(u_i , v_j) + \left\lceil \frac{2f(u_i)f(v_j)}{f(u_i)+f(v_j)} \right\rceil = 1+\left\lceil \frac{2(6n+i-6)(n+j-2)}{7n+i+j-8} \right\rceil = 1+(n+2) \ge (n+3)$$

Case(8): Consider the pair (u_i,w_j), $1 \le i \le n$; $1 \le j \le n-1$

$$d(u_i , w_j) + \left\lceil \frac{2f(u_i)f(w_j)}{f(u_i)+f(w_j)} \right\rceil = 1+\left\lceil \frac{2(6n+i-6)(5n+j- \ )}{11n+i+j-11} \right\rceil = 1+(n+2) \ge (n+3)$$

Case(9): Consider the pair (u_i,x_j), $1 \le i \le n$; $1 \le j \le n-1$

$$d(u_i , x_j) + \left\lceil \frac{2f(u_i)f(x_j)}{f(u_i)+f(x_j)} \right\rceil = 2+\left\lceil \frac{2(6n+i-6)(2n+j-2)}{8n+i+j-8} \right\rceil = 2+(n+2) \ge (n+3)$$

Case(10): Consider the pair (u_i,y_j), $1 \le i \le n$; $1 \le j \le n-1$

$$d(u_i , y_j) + \left\lceil \frac{2f(u_i)f(y_j)}{f(u_i)+f(y_j)} \right\rceil = 3+\left\lceil \frac{2(6n+i-6)(4n+j-4)}{10n+i+j-10} \right\rceil = 1+(n+2) \ge (n+3)$$

Case(11): Consider the pair (u_i,z_j), $1 \le i \le n$; $1 \le j \le n-1$

$$d(u_i , z_j) + \left\lceil \frac{2f(u_i)f(z_j)}{f(u_i)+f(z_j)} \right\rceil = 3+\left\lceil \frac{2(6n+i- \ )(3n+j-3)}{9n+i+j-9} \right\rceil = 2+(n+2) \ge (n+3)$$

Case(12): Consider the pair (v_i,w_j), $1 \le i \le n$; $1 \le j \le n-1$

$$d(v_i , w_j) + \left\lceil \frac{2f(v_i)f(w_j)}{f(v_i)+f(w_j)} \right\rceil = 3+\left\lceil \frac{2(n+i-1)(5n+j-5)}{6n+i+j-6} \right\rceil = 2+(n+2) \ge (n+3)$$

The remaining pairs(v_i,w_j),(v_i,x_j),(v_i,y_j),(v_i,z_j),(w_i,x_j),(w_i,y_j),(w_i,z_j),(x_i,y_j), (x_i,z_j) and (y_i,z_j) also satisfies the radio Harmonic Mean condition. Hence g is a valid radio Harmonic Mean labeling of $Q_n{}^J K_1$. Therefore rhmn $(Q_n{}^J K_1) = 7n - 6$, $n > 2$. □

Example 3.2. Consider the graph $Q_4{}^J K_1$. The radio Harmonic Mean labeling is as shown in figure-2 with rhmn $(Q_4{}^J K_1) = 22$.

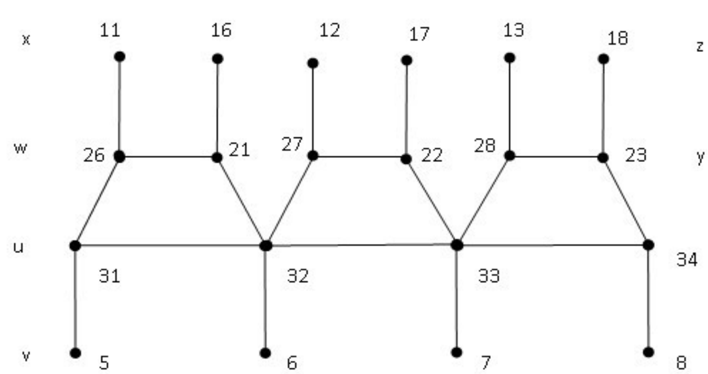

Figure 2. The graph $Q_4{}^J K_1$.

Theorem 3.3. The graph TL_n is a radio Harmonic Mean graph. The radio Harmonic Mean number is $4n - 3$, $n \ge 2$.

Proof. Let TL_n be a triangular ladder. Let $u_1,u_2,\ldots,u_n$ and $v_1,v_2,\ldots,v_n$ be two paths of length n in the graph TL_n. Join $u_{i+1}v_i$, $1 \le i \le n-1$ and join u_iv_i $1 \le i \le n$. The resultant graph is TL_n whose edge set is

$E = \{u_iu_{i+1}, u_{i+1}v_i, v_iv_{i+1}\} \cup \{u_i, v_i\}$ $1 \le i \le n$ and diameter n. Define a function $f : V(TL_n) \rightarrow 1,2,3,\ldots(p + q)$ by $f(u_1) = 1$; $f(u_i) = 2n + i - 3$, $2 \le i \le n$ $f(v_i) = 3n+i-3$, $1 \le i \le n$
now we verify the radio Harmonic Mean condition for f by the condition

$$d(u, v)+ \left\lceil\frac{2f(u)f(v)}{f(u)+f(v)}\right\rceil \ge 1+\text{diam } G$$

Case(1): Consider the pair $(u_i,u_j), i \ne j$; $1 \le i, j \le n$

$$d(u_i ,u_j) + \left\lceil\frac{2f(u_i)f(u_j)}{f(u_i)+f(u_j)}\right\rceil = 1+\left\lceil\frac{2(2n+i-\ \ )(2n+j-2)}{4n+i+j-4}\right\rceil = 1+n \ge 1+\text{diam}(TL_n)$$

Case(2): Consider the pair $(v_i,v_j), i \ne j$; $1 \le i,j \le n$

$$d(v_i , v_j) + \left\lceil\frac{2f(v_i)f(v_j)}{f(v_i)+f(v_j)}\right\rceil = 1+\left\lceil\frac{2(3n+i-3)(2n+j-3)}{6n+i+j-6}\right\rceil = 1+n \ge 1+\text{diam}(TL_n)$$

Case(3): Consider the pair $(u_i,v_j), i \ne j$; $1 \le i,j \le n$

$$d(u_i , v_j) + \left\lceil\frac{2f(u_i)f(v_j)}{f(u_i)+f(v_j)}\right\rceil = 1+\left\lceil\frac{2(2n+i-2)(3n+j-3)}{5n+i+j-5}\right\rceil = 1+n \ge 1+\text{diam}(TL_n)$$

Case(4): Consider the pair (u_{i+1},v_j); $1 \le i,j \le n - 1$

$$d(u_{i+1} , v_j) + \left\lceil\frac{2f(u_{i+1})f(v_j)}{f(u_{i+1})+f(v_j)}\right\rceil = 1+\left\lceil\frac{2(2n+i-2)(3n+j-3)}{5n+i+j-4}\right\rceil = 1+n \ge 1+\text{diam}(TL_n)$$

Hence f is a valid radio Harmonic Mean labeling of TL_n. Therefore $rhmn(TL_n) = 4n - 3, n > 2$.   □
Example 3.3. The following figure shows the radio Harmonic Mean labeling of TL_5, with rhmn = 17.

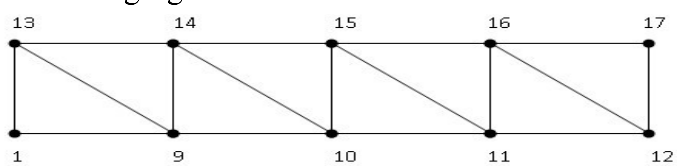

Figure 3. Graph TL_5.

Theorem 3.4. The graph $P_n{}^J k_2$ is a radio Harmonic Mean graph. The radio Harmonic Mean number is $4n - 3$, $n \ge 3$.
Proof. Let $u_1,u_2,\ldots,u_n$ be the path P_n and let v_i,w_i be the vertices of k_2 which are joined to the vertex u_i of path $P_n, 1 \le i \le n$. The resultant graph is $P_n{}^J$ whose edge set is $E = \{u_iu_{i+1}/1 \le i \le n - 1\} \cup \{u_iv_i,u_iw_i/1 \le i \le n\}$ and $\text{diam}(P_n{}^J K_2) = n + 1$.
Define a function f $f : V(P_n{}^J K_2) \rightarrow 1,2,3,\ldots(p + q)$ by $f(u_i) = 3n+i-3$, $1 \le i \le n$
$f(v_i) = n+i-3$, $1 \le i \le n$ $f(w_i) = 2n+i-3$, $1 \le i \le n$
now we verify the radio Harmonic Mean condition for f by the condition

$$d(u, v)+ \left\lceil\frac{2f(u)f(v)}{f(u)+f(v)}\right\rceil \ge 1+\text{diam } G$$

Case(1): Consider the pair $(u_i,u_j), i \ne j$; $1 \le i,j \le n$

$$d(u_i , u_j) + \left\lceil\frac{2f(u_i)f(u_j)}{f(u_i)+f(u_j)}\right\rceil = 1+\left\lceil\frac{2(3n+i-3)(3n+j-\ \ )}{6n+i+j-6}\right\rceil = 1+(n+1) \ge 1+\text{diam}(P_n \Theta k_2)$$

Case(2): Consider the pair $(v_i,v_j), i\ne j$; $1 \le i,j \le n$

$$d(v_i , v_j) + \left\lceil\frac{2f(v_i)f(v_j)}{f(v_i)+f(v_j)}\right\rceil = 3+\left\lceil\frac{2(n+i-3)^2}{2n+i+j-6}\right\rceil = 1+(n+1) \ge 1+\text{diam}(P_n \Theta k_2)$$

Case(3): Consider the pair $(w_i,w_j), i\neq j;$ $1 \leq i,j \leq n$

$$d(w_i\,,\,w_j) + \left\lceil\frac{2f(w_i)f(w_j)}{f(w_i)+f(w_j)}\right\rceil = 3+\left\lceil\frac{2(2n+i-3)^2}{4n+i+j-6}\right\rceil = 1+(n+1) \geq 1+diam(P_n \Theta k_2)$$

Case(4): Consider the pair $(u_i,v_j), i\neq j;$ $1 \leq i,j \leq n$

$$d(u_i\,,\,v_j) + \left\lceil\frac{2f(u_i)f(v_j)}{f(u_i)+f(v_j)}\right\rceil = 3+\left\lceil\frac{2(2n+i-3)(3n+i-3)}{5n+i+j-6}\right\rceil = 1+(n+1) \geq 1+diam(P_n \Theta k_2)$$

Case(5): Consider the pair $(u_i,w_j), i\neq j;$ $1 \leq i,j \leq n$

$$d(u_i\,,\,w_j) + \left\lceil\frac{2f(u_i)f(w_j)}{f(u_i)+f(w_j)}\right\rceil = 1+\left\lceil\frac{2(2n+i-3)(3n+i-3)}{5n+i+j-6}\right\rceil = 1+(n+1) \geq 1+diam(P_n \Theta k_2)$$

Case(6): Consider the pair $(v_i,w_j), i\neq j;$ $1 \leq i,j \leq n$

$$d(v_i\,,\,w_j) + \left\lceil\frac{2f(v_i)f(w_j)}{f(v_i)+f(w_j)}\right\rceil = 2+\left\lceil\frac{2(2n+i-3)(2n+j-3)}{3n+i+j-6}\right\rceil = 1+(n+1) \geq 1+diam(P_n \Theta k_2)$$

Thus the radio Harmonic Mean condition is satisfied for all pairs of vertices. Hence f is a valid radio Harmonic Mean labeling of $P_n{}^J k_2$. Hence rhmn$(P_n{}^J k_2)$ =i
$4n-3, n \geq 3.$ □

Example 3.4. The following figure shows the radio Harmonic Mean labeling of $P_n{}^J k_2$, with rhmn = 9.

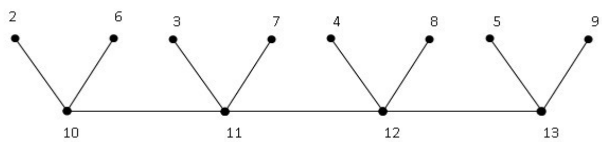

Figure 4. Graph $P_n{}^J k_2$.

ACKNOWLEDGEMENT:

The authors are thankful to the referees for their valuable suggestions and fruitful comments.

REFERENCES :

1. P. S. Bullen, Handbook of means and their inequalities, Kluwer Acad. Publ., Dordrecht, 2003.
2. J.A. Gallian, A Dynamic Survey of Graph Labeling. The Electronic Journal of combinatorics, 2013.
3. Harary. F, Graph Theory, Narosa publishing House, New Delhi, (1988).
4. C. Jayasekaran and C. David Raj, Harmonic Mean labeling of disconnected graphs, Journal of Discrete Mathematical Sciences and Cryptography, 19(1), (2016), pp. 1 - 12.
5. R Sampath Kumar, G Narasimhan and R Chandrasekhar, Contra Harmonic Mean labeling of graph, Mapana J Sci, 12(3), 2013, pp. 23 - 29.
6. Sampath Kumar R, Narasimhan G and K M Nagaraja, Heron mean labeling of graph, International Journal of Recent Scientific Research, 8(9), 2017, pp. 19808 - 19811.
7. Sampath Kumar R, and K M Nagaraja, Centroidal Mean Labeling of graphs, International Journal of Research in Advent Technology, 7(1), 2019, pp. 183 - 189.
8. S. Somasundaram and P Vidhyarani, Geometric Mean labeling of graphs, Bulletin of Pure and Applied Sciences, 30(2), 2011, pp. 153 - 160.
9. S. Somasundaram and R.Ponraj, Mean Labeling of graphs, National Academy of Science Letters (1988), 26 (2003), pp. 210 - 213.
10. R. Sampath Kumar, Narasimhan G, Chandrasekar R, Contra Harmonic Mean labelling of graphs, Mapana Journal of Sciences., 12(3), 2013, 23-29. doi:10.12723/mjs.26.4. ISSN: 0975-3303