The stochastic modelling for climate change: An applied mathematics perspective modelling

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Abstract: Here we describe a systematic strategy from applied mathematics for probabilistic climate modeling. One of the topics discussed was the probabilistic modeling of mid-latitude low-frequency variations due to several communication patterns, including the central role and physical mechanisms responsible for multiplicative noise. Here, we develop a new low-dimensional stochastic model that mimics key features of general circulation models (GCMs) and test the accuracy of stochastic modal reduction methods. The second topic discussed here is the systematic design of probabilistic mesh models to capture irregular and highly intermittent features that are not resolved by deterministic parameterization. A recently applied mathematical construction principle for discontinuous stochastic sequence modeling is demonstrated in the idealized environment of deep tropical convection. The practical effect of this stochastic model on both slowing convective coupled waves and increasing jitter is shown here.

Keywords: Low-Frequency Variability, Tropical Convection, Multiplicative Noise, Intermittency

1. Introduction

Stochastic modelling for climate is important for understanding the intrinsic variability of dominant low-frequency teleconnection patterns in climate, to provide cheap low-dimen- sional computational models for the coupled atmosphere-ocean system, and to reduce model error in standard deterministic computer models for extended-range prediction through appropriate stochastic noise (Palmer 2001).

The present contribution is a research-expository paper on systematic strategies for stochastic climate modelling from the perspective of modern applied mathematics. In the modern applied mathematics 'modus operandi' rigorous mathematical analysis, qualita- tive, asymptotic, and numerical modelling are all blended together in a multi-disciplinary fashion to provide systematic guidelines to address real world problems (Majda 2000). For stochastic modelling in climate, the modern applied mathematics tool kit includes stochas- tic differential equations and discontinuous Markov jump processes (Gardiner 1985), sys- tematic asymptotic reduction techniques, nonlinear dynamical systems theory, and ideas from both statistical physics (Majda & Wang 2006) and mathematical statistics (Kravtsov *et al.* 2005; Majda *et al.* 2006a); mathematical rigour provides unambiguous guidelines in idealised models. Another facet of the modern applied mathematics philosophy is the development of qualitative models which represent a Platonic ideal for central issues simul- taneously in diverse scientific disciplines such as material science, biomolecular dynamics, and climate science.

In section 2 below we illustrate and apply this modern applied mathematics philosophy to stochastic modelling of the low-frequency variability of the atmosphere. The systematic mathematical theory (Majda *et al.* 1999, 2001, 2002, 2003, 2006b; collectively referred to as MTV hereafter; Franzke *et al.* 2005, Franzke & Majda 2006) for these problems is briefly reviewed including the central role and physical mechanisms responsible for mul- tiplicative noise in the low-frequency dynamics. In this context, the Platonic ideal from applied mathematics is the truncated Burgers-Hopf model (Majda & Timofeyev 2000). A new simplified low-dimensional stochastic model which reproduces key features of atmo- spheric GCM's is utilised there to test the fidelity of stochastic mode reduction techniques. A recent diagnostic statistical test with firm mathematical underpinning for understand- ing and interpreting the dynamical sources of the small departures from Gaussianity in low-frequency variables (Franzke *et al.* 2007) is also developed there.

While section 2 deals with applied mathematical modelling through stochastic differ- ential equations, section 3 is devoted to the systematic development of stochastic lattice models to capture unresolved features that are highly intermittent in space and time such as deep convective clouds, cloud cover in sub-tropical boundary layers, sub-mesoscale eddies in the ocean, and mesoscale sea-ice cover. Here the mathematical tools involve a family of discontinuous Markov jump processes with multi-scale behaviour in space-time called stochastic spin-flip models. The key mathematical development involves systematic strategies to coarse grain such stochastic spin-flip models to achieve computational effi- ciency while retaining crucial features of the microscale interactions (Katsoulakis *et al.* 2003a, 2003b; Katsoulakis & Vlachos 2003). The use of such stochastic lattice models to parametrise key features of tropical convection is briefly reviewed (Majda & Khouider 2002; Khouider *et al.* 2003). For the coupling of continuum models like a GCM to a stochastic lattice model as well as in many diverse applications, an applied mathemat- ics Platonic ideal model has recently been introduced and analysed by Katsoulakis *et al.*
(2004, 2005, 2006, 2007; hereafter KMS). This model consists of a system of ODE's for continuum variables $\dot{X}$,

$$\frac{d\dot{X}}{dt} = \dot{F}(\dot{X}, \bar{\sigma}) \tag{1.1}$$

two-way coupled to a stochastic spin-flip model written abstractly here as

$$\frac{d}{dt}\mathsf{E}f(\sigma) = \mathsf{E}Lf(\sigma) \tag{1.2}$$

where σ denotes the spatial coverage, L is the generator, f is a test function, and E de- notes the expected value. This idealised class of models has been utilised to systematically analyse the effects of various coarse-graining procedures on processes with intermittency, large-scale bifurcations, and microscale phase transitions (KMS 2004, 2005, 2006, 2007). A concrete example for tropical convection in climate is given in section 3. A new appli- cation of these stochastic lattice models to capture intermittent features and improve the fidelity of deterministic parametrisations of convection with clear deficiencies is developed in section 3. First the systematic design principles for (1.1) and (1.2) (KMS 2006, 2007) are utilised to calibrate a stochastic column model for tropical convection with intermit- tency; then new results are presented on the practical effect of slowing down convectively coupled waves and increasing their fluctuations through the stochastic lattice models.

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2. Systematic Low Dimensional Stochastic Mode Reduction and Atmospheric Low-Frequency Variability

A remarkable fact of Northern Hemisphere low-frequency variability is that it can be ef- ficiently described by only a few teleconnection patterns which explain most of the total variance (e.g. Wallace & Gutzler 1981). These few teleconnection patterns not only exert a strong influence on regional climate and weather, they are also related to climate change (Hurrell 1995). These properties of teleconnection patterns make them an attractive choice as basis functions for climate models with a highly reduced number of degrees of free- dom. The development of such reduced climate models involves the solution of two major issues: (1) How to properly account for the unresolved modes, also known as the closure problem; and (2) How to define a small set of basis functions which optimally represent the dynamics of the major teleconnection patterns. This section addresses primarily issue
(1) and presents a rigorous strategy of how to systematically account for the unresolved degrees of freedom.

The simplest approach to derive highly truncated models of teleconnection patterns is to empirically fit simple stochastic models (e.g. autoregressive models and fractionally dif- ferenced models) to individual scalar teleconnection indices (Feldstein 2000; Stephenson *et al.* 2000; Percival *et al.* 2001). Statistical tests usually cannot distinguish if short- or long-memory models provide the better fit. A more complex approach, which also tries to capture deterministic interactions between different teleconnection patterns, is to linearise the equations of motion around a climatological mean state. Such models can be deter- mined empirically from data or by using the linearised equations of motion. These models can either be forced by a random forcing (Branstator 1990; Newman *et al.* 1997; Whitaker & Sardeshmukh 1998; Zhang & Held 1999) or by an external forcing representing tropical heating (Branstator & Haupt 1998). To ensure stability of these linear models damping is added according to various ad hoc principles. There is a recent survey of such modelling strategies (Delsole 2004).

A more powerful method is to empirically fit nonlinear stochastic models with possi- bly multiplicative (state dependent) noise by using the Fokker-Planck equation (Gardiner 1985; Sura 2003; Berner 2005). To reliably estimate the drift and diffusion coefficients in the Fokker-Planck equation is a subtle inverse problem which requires very long time series and is further complicated by the need to retain the leading order eigenvalue struc- ture of the Fokker-Planck operator in order to keep the autocorrelation time scales of the original model (Crommelin & Vanden-Eijnden 2006); the fitting procedure in this most recent work is the most attractive current regression strategy for low-frequency behaviour. Recently Kravtsov *et al.* (2005) have developed a simplified nonlinear regression strategy which produces very good results for a three-layer quasi-geostrophic model with a realistic climate. However, order 2000 regression coefficients need to be fitted in a model with order 1000 state variables to achieve these results. Some inherent limitations of this approach in describing the correct physics are discussed briefly below in a simplified model.

All the work presented above derives reduced models by regression fitting of the re- solved modes. Another approach is to take advantage of the basis function property of teleconnection patterns. Schubert (1985), Selten (1995) and Achatz & Branstator (1999) developed low-order models with EOFs as basis functions. Truncated EOF models expe- rience climate drift due to the neglected interactions with the unresolved modes. Selten (1995) and Achatz & Branstator (1999) parameterise these neglected interactions by a lin- ear damping, whose strength is determined empirically. A possibly more powerful tool to

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represent the dynamics of a system are Principal Interaction Patterns (PIP; Hasselmann 1988; Kwasniok 1996, 2004). The calculation of PIPs takes into account the dynamics of the model for which one tries to find an optimal basis and also often involves ad hoc clo- sure through linear damping and an ansatz for nonlinear interactions. Crommelin & Majda (2004) compare different optimal bases. They find that models based on PIPs are superior to models based on EOFs. On the other hand, they also point out that the determination of PIPs can show sensitivities regarding the calculation procedure, at least for some low- order atmospheric dynamical systems with regime transitions. This feature can make PIPs possibly a less attractive basis.

Majda *et al.* (1999, 2001, 2002, 2003, 2005, 2006b) provide a systematic framework for how to account for the effect of the fast degrees of freedom on the slow modes in com- bination with using the dominant teleconnection patterns as basis functions. In contrast to the empirical fitting procedures applied in the studies discussed above the stochastic mode reduction strategy put forward in MTV *predicts* the functional form of all deterministic and stochastic correction terms and provides a *minimal* regression fitting procedure of only the *fast modes* (Franzke *et al.* 2005; Franzke & Majda 2006). In general only an estimate for the variance and eddy turnover time for each fast mode is needed. It has been applied and tested on a wide variety of simplified models and examples.

(a) Overview of the MTV Strategy

We illustrate the ideas for stochastic climate modelling by considering the following prototype equation for geophysical flow:

$$\frac{\partial u}{\partial t} = F + Lu + B(u, u). \tag{2.1}$$

position through the variables $\tilde{u}$ and u' which are characterised by strongly differing time scales (MTV 1999, Majda *et al.* 2005). The variable $\tilde{u}$ denotes a slow low-frequency mode In stochastic climate modelling, the variable u is decomposed into an orthogonal decom-
to the u' variables (also referred to as fast mode). By decomposing $u = \tilde{u} + u'$ in terms of (also referred to as climate mode) of the system, which evolves slowly in time compared some optimal energy norm basis we can write them as

$$u = \sum_{i=1}^{N} a_i \mathbf{e}_i = \sum_{i=1}^{R} \alpha_i \mathbf{e}_i + \sum_{j=R+1}^{N} \beta_j \mathbf{e}_j, \tag{2.2}$$

with $\tilde{u} = \sum_{i=1}^{R} \alpha_i \mathbf{e}_i$, and $u' = \sum_{j=R+1}^{N} \beta_j \mathbf{e}_j$, where R is the number of climate modes, a_i denote the expansion coefficients, α_i (β_j) are the expansion coefficients of the slow (fast) modes. The use of the energy norm ensures the conservation of energy by the nonlinear operator (Selten 1995). By properly projecting the energy norm basis, derived from the geophysical model, onto Eq. (2.1), we get two sets of equations for slow α_i and fast β_i

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modes:

$$\dot{\alpha}_i(t) = \varepsilon H_i^{\alpha} + \sum_j L_{ij}^{\alpha\alpha}\alpha_j(t) + \frac{1}{\varepsilon}\sum_j L_{ij}^{\alpha\beta}\beta_j(t) + \sum_{jk} B_{ijk}^{\alpha\alpha\alpha}\alpha_j(t)\alpha_k(t) + \frac{2}{\varepsilon}\sum_{jk} B_{ijk}^{\alpha\alpha\beta}\alpha_j(t)\beta_k(t) + \frac{1}{\varepsilon}\sum_{jk} B_{ijk}^{\alpha\beta\beta}\beta_j(t)\beta_k(t) \quad (2.3)$$

$$\dot{\beta}_i(t) = \varepsilon H_i^{\beta} + \frac{1}{\varepsilon}\sum_j L_{ij}^{\beta\alpha}\alpha_j(t) + \frac{1}{\varepsilon}\sum_j L_{ij}^{\beta\beta}\beta_j(t) + \frac{1}{\varepsilon}\sum_{jk} B_{ijk}^{\beta\alpha\alpha}\alpha_j(t)\alpha_k(t) + \frac{2}{\varepsilon}\sum_{jk} B_{ijk}^{\beta\alpha\beta}\alpha_j(t)\beta_k(t) + \frac{1}{\varepsilon^2}\sum_{jk} B_{ijk}^{\beta\beta\beta}\beta_j(t)\beta_k(t), \quad (2.4)$$

where the nonlinear operators have been symmetrised, i.e. $B_{ijk} = B_{ikj}$ in (2.3) and (2.4). The upper indices α and β indicate the respective subsets of the full operators in (2.1). Here

ε is a small positive parameter which controls the separation of time scale between slow and fast modes and measures the ratio of the correlation time of the slowest non-climate mode

u^J to the fastest climate mode $\tilde{u}$ In placing the parameter in front of particular terms we

modes alone. Ultimately, ε is set to the value $\varepsilon = 1$ in developing all the final results tacitly assume that they evolve on a faster time scale then the terms involving the climate (MTV 2002, 2003), i.e. introducing ε is only a technical step in order to carry out the MTV

mode reduction strategy. Such a use of ε has been checked on a wide variety of idealised examples where the actual value of ε ranges from quite small to order one (MTV 2002, 2003, 2006b; Majda & Timofeyev 2004). Following MTV (1999, 2001, 2002, 2003, 2006b) and Franzke *et al.* (2005) the mode elimination procedure is based on the assumption that the dynamics of the fast modes alone in Eq. (2.4), i.e. the dynamical system

$$\dot{c}_i = \sum_{jk} B_{ijk}^{\beta\beta\beta} c_j c_k, \quad (2.5)$$

is ergodic and mixing with integrable decay of correlation. In other words, we assume that for almost all initial conditions, and suitable function f and g, we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(c(t))\,dt = \langle f \rangle \quad (2.6)$$

where $\langle\cdot\rangle$ denotes expectation with respect to some appropriate invariant distribution, and

$$G(s) = \lim_{T\to\infty}\frac{1}{T}\int_0^T g(c(t+s),c(t))\,dt - \lim_{T\to\infty}\frac{1}{T^2}\int_0^T\int_0^T g(c(t),c(t'))\,dt\,dt' \quad (2.7)$$

is an integrable function of s, i.e. $\int_0^\infty |G(s)|\,ds < \infty$. Furthermore, we assume that the it can be shown in the limit $\varepsilon \to 0$ (Kurtz 1973; MTV 2001) that the dynamics of the low order statistics for the fast modes in (2.5) are Gaussian. Under the above assumptions, slow modes α_i in (2.3) can be written as the following Ito stochastic equation for the slow

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modes alone:

$$
\begin{aligned}
d\alpha_i(t) = {} & \lambda_B \left(H_i^{\alpha} dt + \sum_j L_{ij}^{\alpha\alpha}\alpha_j(t)dt + \sum_{jk} B_{ijk}^{\alpha\alpha\alpha}\alpha_j(t)\alpha_k(t)dt \right) \\
& + \lambda_A^2 \sum_j \tilde{L}_{ij}^{(2)}\alpha_j(t)dt + \lambda_A \sqrt{2}\sum_j \sigma_{ij}^{(2)} dW_j^{(2)} \\
& + \lambda_M^2 \left(\sum_j \tilde{L}_{ij}^{(3)}\alpha_j(t)dt + \sum_{jkl} \tilde{M}_{ijkl}\alpha_j(t)\alpha_k(t)\alpha_l(t)dt \right) \\
& + \lambda_L^2 \left(\sum_j \tilde{L}_{ij}^{(1)}\alpha_j(t)dt \right) \\
& + \lambda_M\lambda_L \left(\tilde{H}_i^{(1)} dt + \sum_{jk} \tilde{B}_{ijk}\alpha_j(t)\alpha_k(t)dt \right) \\
& + \lambda_A\lambda_F \tilde{H}_i^{(2)} dt \\
& + \sqrt{2}\sum_j \sigma_{ij}^{(1)}(\alpha(t))dW_j^{(1)},
\end{aligned}
\tag{2.8}
$$

where the nonlinear noise matrix $\sigma^{(1)}$ satisfies,

$$
\lambda_L^2 Q_{ij}^{(1)} + \lambda_L\lambda_M \sum_k U_{ijk}\alpha_k(t) + \lambda_M^2 \sum_{kl} V_{ijkl}\alpha_k(t)\alpha_l(t) = \sum_k \sigma_{ik}^{(1)}(\alpha(t))\sigma_{jk}^{(1)}(\alpha(t)).
\tag{2.9}
$$

It is guaranteed (MTV 2001) that the operator on the left hand side of (2.9) is always pos- itive definite ensuring the existence of the nonlinear noise matrix on the right hand side. All coefficients are defined explicitly in MTV (2001) and Franzke *et al.* (2005). A com- prehensive mathematical theory of the stochastic mode reduction strategy for geophysical applications is developed in MTV (2001) with many new mathematical phenomena in the resulting equations explored there.

To see which of these correction terms play a vital role in the integrations of the low- order stochastic model (2.8) we grouped the interaction terms between slow and fast modes according to their physical origin and set a parameter λ_i in front of the corresponding in- teraction coefficient. The bare truncation is indicated by a λ_B and describes the interaction between the slow modes. The interaction between the triads $B^{\alpha\beta\beta}$ and $B^{\beta\alpha\beta}$ gives rise to additive noise and a linear correction term and arises from the advection of the fast modes by the slow ones; we name these triads “additive” triads and set a λ_A in front of them (MTV 1999, 2001, 2002, 2003). The second type of triad interaction is between $B^{\alpha\alpha\beta}$ and $B^{\beta\alpha\alpha}$. These interactions create multiplicative noises and cubic nonlinear correction terms (MTV 1999, 2001, 2002, 2003); we call them “multiplicative” triads in the following and indicate them by a λ_M . These triad interactions describe the advection of slow modes by the fast modes which induce tendencies in the slow modes. The linear coupling between the slow and fast modes $L^{\alpha\beta}$, and $L^{\beta\alpha}$, give rise to additive noise and a linear correction term (MTV 2001; Franzke *et al.* 2005), which is called the augmented linearity here and is indicated by a λ_L. The augmented linearity describes the effect of the linear interaction be- tween the fast (slow) modes and the climatological mean state onto the slow (fast) modes and is the main interaction captured in the linear stochastic modelling strategy (Delsole 2004). We set a λ_F in front of the last remaining interaction term $L^{\beta\beta}$, the linear coupling of the fast modes. The quadratic nonlinear corrections, a forcing term and a further multi- plicative noise contribution are caused by the interaction between the linear coupling terms and the "multiplicative" triads. Another forcing correction term comes from the interaction between "additive" triads and the linear coupling of the fast modes.

In chapter 3 of Majda *et al.* (2005) a simplified three mode elementary "toy climate model" is discussed and the MTV procedure is applied explicitly to that example. The origin of all the terms in (2.8) is developed in a transparent fashion in these examples. Once the low-order stochastic model has been developed from the above procedure, one can asses the importance of the various deterministic and stochastic processes systemati- cally by varying the coefficients λ_B, λ_A, λ_L, λ_M and λ_F systematically in (2.8) (Franzke *et al.* 2005) and even develop simple physically motivated regression fitting strategies (Franzke & Majda 2006). In interesting recent work, Sura & Sardeshmukh (2007) have utilised scalar linear stochastic models with multiplicative and additive noise to explain non-Gaussian SST variability. If the reduced stochastic models in (2.8) from the MTV pro- cedure are linearised at the climate mean state, they automatically produce vector systems of linear stochastic equations with both multiplicative and additive noise with the same structure, with clear sources for the underlying physical contributions to this equation.

(b) Idealised Models for Stochastic Mode Reduction

The idealised models, where the procedure has been tested, have order 100 degrees of freedom and include those with trivial climates (MTV 2002), periodic orbits or multiple equilibria (MTV 2003), and heteroclinic chaotic orbits coupled to a deterministic bath of modes satisfying the truncated Burgers-Hopf equation (Majda & Timofeyev 2000, 2004). The truncated Burgers-Hopf equation is a toy model with some remarkable features mimic- ing behaviour in the real atmosphere; it has a well defined equipartition spectrum and a sim- ple scaling theory for correlations with the large scales decorrelating more slowly than the small scales, i.e. low-frequency variability. Furthermore, these predictions are confirmed with very high precision by numerical simulations (Majda & Timofeyev 2000; Majda & Wang 2006). The MTV procedure has been validated in these examples even when there is little separation of time scales between slow and fast modes. In the example of a four di- mensional resonant system with chaotic dynamics coupled to the truncated Burgers system, only one empirical regression fitting coefficient is used and complex bifurcation diagrams and PDF's in a climate change scenario are reproduced by the four dimensional stochastic mode reduction resulting from the MTV procedure applied to the 104 degree of freedom deterministic system. An especially stringent recent test is the application of this proce- dure to the first few large scale modes of the truncated Burgers equations in the turbulent cascade (MTV 2006b).

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(c) A priori stochastic modelling for mountain torque

The ideal barotropic quasi-geostrophic equations with a large scale zonal mean flow U on a $2\pi \times 2\pi$ periodic domain (Carnevale & Frederiksen, 1987) are given by

$$\frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q + U\frac{\partial q}{\partial x} + \beta\frac{\partial \psi}{\partial x} = 0$$
$$q = \Delta\psi + h \qquad (2.10)$$
$$\frac{dU}{dt} = \frac{1}{4\pi^2}\int h\frac{\partial \psi}{\partial x}dxdy$$

with q the potential vorticity, U the large scale zonal mean flow, ψ the stream function, and h the topography. In (2.10), the mean flow changes in time through the topographic stress; this effect is the direct analog for periodic geometry of the change in time of an- gular momentum due to mountain torque in spherical geometry (Frederiksen *et al.* 1996; Madja & Wang 2006). Here the a priori stochastic modelling strategy (MTV 1999, 2001) is applied to the stochastic modelling of the topographic stress terms in (2.10) as an analog for mountain torque; thus, the variable, U, is the slow variables while all the modes ψ_k are fast variables for the MTV procedure.

In this example the systematic stochastic modelling procedure (MTV 2003) results in the predicted nonlinear reduced equation for U,

$$\frac{dU}{dt} = -\gamma(U)U + \frac{\gamma'(U)}{\alpha\mu} + \sqrt{\frac{2\gamma(U)}{\alpha\mu}}\dot{W} \qquad (2.11)$$

where $\gamma'(U) = d\gamma/dU$ and

$$\gamma(U) = 2\sum_{\boldsymbol{k}} \frac{\mu k_x^2\, |H_{\boldsymbol{k}}|^2\, \gamma_{\boldsymbol{k}}}{\gamma_{\boldsymbol{k}}^2 + (\Omega_{\boldsymbol{k}} - k_x U)^2}. \qquad (2.12)$$

Here $\Omega_{\boldsymbol{k}} = \frac{k_x\beta}{|\boldsymbol{k}|^2} - Uk_x$ is the Rossby wave frequency Doppler shifted by the mean flow.

Under the additional assumption that $|k_x U|^2 \ll \gamma_{\boldsymbol{k}}^2 + (\Omega_{\boldsymbol{k}})^2$, a standard predicted linear stochastic model for U emerges from (2.11) with $\gamma = \gamma(0)$ from (2.12) and $\gamma'(0) = 0$ (MTV 1999, 2001). It is shown in MTV 2003 that this nonlinear stochastic equation is superior to the linear one for large amplitude topography where $\varepsilon \cong 0.7$. This is the sim- plest example with multiplicative noise. Egger (2005) has utilised the systematic strategies from MTV (1999, 2003) to improve regression strategies for analysing observational data for angular momentum. In Majda *et al.* (2006a), the recent systematic regression fitting strategy mentioned earlier (Crommelin & Vanden-Eijnden 2006) is applied to (2.10) and independently confirms the predictions in (2.11) and (2.12).

(d) Geophysical and Climate Models

Franzke *et al.* (2005) put the above systematic stochastic mode reduction strategy in a form which makes the practical implementation of the MTV procedure in complex geo- physical models simpler with the same reduced stochastic equations for the fast modes. In this study a T21 truncated barotropic model on the sphere with a realistic climate was used to derive low-order stochastic models by the MTV strategy. Low-order models with as little as 2 slow modes succeed in capturing the geographical distributions of the clima- tological mean field, the variance and the eddy forcing.

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Recently, Franzke & Majda (2006) applied the systematic stochastic mode reduction strategy to a baroclinic three layer quasi-geostrophic model on the sphere (Marshall and Molteni 1993) which mimics the climatology of the European Centre for Medium-Range Weather Forecasts (ECMWF) reanalysis data. The low-order stochastic climate model con- sists of the climate modes as slow modes defined as the leading total energy norm EOFs and the stochastic mode reduction procedure *predicts* all forcing, linear, quadratic and cu- bic correction terms as well as additive and multiplicative noises; these correction terms and noises account for the interaction of the climate modes with the neglected non-climate modes and the self-interaction amongst the non-climate modes. For the three layer quasi-geostrophic model low-order stochastic models with 10 or less climate modes reproduce the geographical distributions of the standard deviation and eddy forcing well. They un- derestimate the standard deviations by at most a factor of about 1.5. Furthermore, they re- produce the autocorrelation functions reasonably well. A budget analysis shows that both linear and nonlinear correction terms as well as both additive and multiplicative noises are important. The physical intuition behind the noises as derived from the MTV proce- dure is as follows: The additive noise stems from the linear interaction between the fast modes and the climatological mean state, and the multiplicative noise comes from the ad- vection of the slow modes by the fast modes. All these deterministic correction terms and noises (both additive and multiplicative) are *predicted* by the systematic stochastic mode reduction strategy, whereas, previous studies a priori approximate the nonlinear part of the equations by a linear operator and additive noise. This noise is typically white in time but may be spatially correlated. In other words, these studies truncate the dynamics on both the slow and fast modes, and add ad hoc damping in order to stabilise the linear model (Whitaker & Sardeshmukh 1998; Zhang & Held 1999). The systematic MTV approach summarised briefly below truncates the dynamics only on the fast modes and predicts the functional form of all necessary nonlinear correction terms and noises; therefore, it also predicts the necessary damping.

The MTV stochastic climate models for this application experience some climate drift. A minimal empirical MTV model without climate drift can be constructed through three parameter regression fitting by down scaling the bare truncation terms and up scaling the two important MTV processes (augmented linearity and multiplicative triads). These em- pirical MTV stochastic climate models with minimal regression fitting still capture the geographical distribution of the standard deviation and eddy forcing and the autocorrela- tion functions reasonably well, while not experiencing climate drift. This surprising result can be interpreted as the fact that the climate modes are predominantly driven by the fast modes and the self-interactions among the slow modes are less important, as can already be seen from the bare truncation models, which do not capture any feature of the actual dynamics. Furthermore, these empirical MTV stochastic climate models suggest that the bare truncation is likely the cause of the climate drift. Integrations of bare truncation mod- els (without any MTV correction terms) already produce a big climate drift (Franzke and Majda 2006). The MTV mode reduction procedure is able to reduce the climate drift in most of the slow modes, but is not able to overcome it completely. Previous results with a variety of simplified models show no climate drift in a MTV framework (MTV 1999, 2001, 2002, 2003; Majda & Timofeyev 2004). This is likely because, these simplified models are constructed in such a way that they have an optimal basis which captures the dynamics of the climate modes. This gives evidence that total energy norm EOFs are not an ade- quate dynamical basis in capturing the dynamics of the slow modes. Further details of this application can be found in Franzke & Majda (2006).

(e) A Simple Stochastic Model with Key Features of Atmospheric Low-Frequency Variability

In this section we present a 4 mode stochastic climate model of the kind put forward in Majda *et al.* (2005; Chapter 3). This simple stochastic climate model is set up in such a way that it features many of the important dynamical features of comprehensive GCMs but with many fewer degrees of freedom. Such simple toy models allow the efficient exploration of the whole parameter space which is impossible to conduct with GCMs. Thus, we are able to test the predictions of the above framework with direct model experiments by switching on and off certain terms rather than relying on diagnostic methods.

While this model is not rigorously derived from a geophysical flow model (e.g. barotropic vorticity equation), it has the same functional form one would end up with when deriving a reduced stochastic model from a geophysical model. Thus, consistent with geophysical flow models the toy model has a quadratically nonlinear part which conserves energy, a linear operator, and a constant forcing, which in a geophysical model would represent the effects of external forcing such as solar insulation and sea surface temperature. The lin- ear operator has two contributions: One is a skew-symmetric part formally similar to the Coriolis effect and topographic Rossby wave propagation. The other is a negative definite symmetric part formally similar to dissipative processes such as surface drag and radiative damping.

The model is constructed in such a way that there are two modes, denoted by **x**, that evolve more slowly than the other two modes, **y**. In realistic models there would be very many additional fast modes representing e.g. synoptic weather systems or convection. To mimic their combined effect, we include damping and stochastic forcing $-\frac{\gamma}{\varepsilon}\mathbf{y} + \frac{\sigma}{\sqrt{\varepsilon}}\dot{\mathbf{W}}$ in the equations for **y** where **W** denotes a Wiener process. The motivation for this approx- imation is that these fast modes are associated with turbulent energy transfers and strong mixing and that we do not require a more detailed description since we are only interested in their effect on the slow modes. The two fast modes carry most of the variance in this model but as noted earlier, these two modes are surrogates in the model for the entire bath of fast modes so this is very natural. The parameter ε controls the time scale separation between the slow and fast variables. For testing the predictions of the general framework that is derived in the previous section, we will treat the two slow modes **x** as the climate modes and the two fast modes as the non-climate modes **y**. Therefore, our toy model has the following form

$$dx_1 = ((-x_2(L_{12} + a_1x_1 + a_2x_2) + d_1x_1 + F_1) + L_{13}y_1 + b_{123}x_2y_1)\,dt \tag{2.13a}$$

$$dx_2 = ((+x_1(L_{21} + a_1x_1 + a_2x_2) + d_2x_2 + F_2) + L_{24}y_2 + b_{213}x_1y_1)\,dt \tag{2.13b}$$

$$dy_1 = \left(-L_{13}x_1 + b_{312}x_1x_2 + F_3 - \frac{\gamma_1}{\varepsilon}y_1\right)dt + \frac{\sigma_1}{\sqrt{\varepsilon}}\,dW_1 \tag{2.13c}$$

$$dy_2 = \left(-L_{24}x_2 + F_4 - \frac{\gamma_2}{\varepsilon}y_2\right)dt + \frac{\sigma_2}{\sqrt{\varepsilon}}\,dW_2 \tag{2.13d}$$

following relation: $b_{123} + b_{213} + b_{312} = 0$ while the nonlinear bare truncation terms also To ensure energy conservation of the nonlinear operator the coefficients have to satisfy the conserve energy. In this particular set up the slow modes and the fast modes are coupled

through two mechanisms; one is a skew-symmetric linear coupling and the other is nonlin- ear triad interaction. The nonlinear coupling involving b_{ijk} produces multiplicative noise in the MTV framework (MTV 1999, 2003; see also Eq. (2.15)) so we refer to it as a multi-

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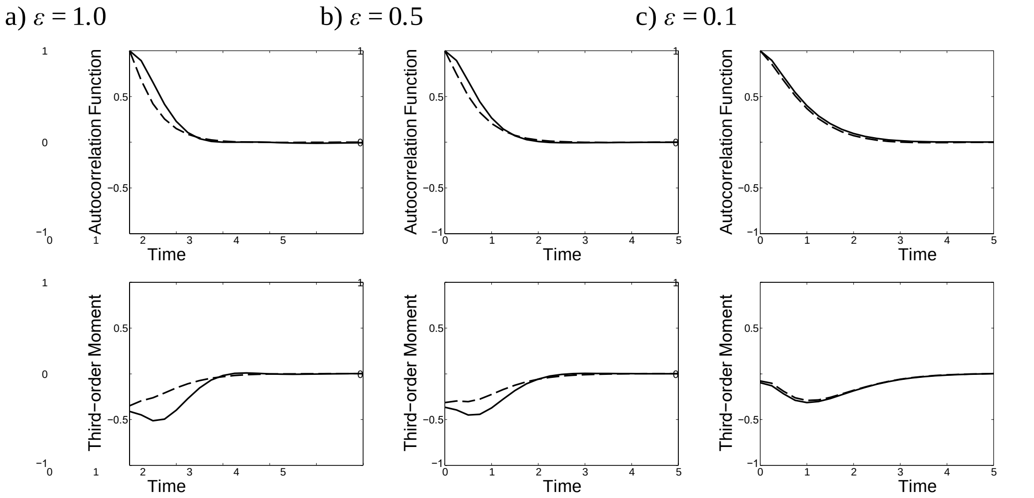

Figure 1. Autocorrelation function and third-order moment for different values of ε. Solid line: stochastic dynamics; dashed line: reduced dynamics.

as ε 0 is done explicitly through elementary manipulations (MTV 1999, 2001; Majda *et* plicative triad. One advantage of the model in (2.13) is that the stochastic mode reduction *al.* 2005). The corresponding reduced Itô SDE for the climate variables alone is given by

$$dx_1(t) = (-x_2(t)(\omega + a_1x_1(t) + a_2x_2(t)) + d_1x_1(t) + F_1)\,dt + \frac{\varepsilon}{\gamma_1}(L_{13}F_3 - L_{13}L_{13}x_1(t) + b_{123}F_3x_2(t) + L_{13}b_{312}x_1(t)x_2(t) - L_{13}b_{123}x_1(t)x_2(t) + b_{312}b_{123}x_1(t)x^2(t)\Big)dt \;\; 2$$

$$\frac{\sigma^2}{2} + \varepsilon\frac{1}{\gamma}b_{213}b_{123}x_1(t)dt$$

$$\sqrt{\varepsilon}\frac{\sigma_1}{\gamma_1}(L_{13} + b_{123}x_2(t))dW_1(t) \tag{2.14a}$$

$$dx_2(t) = (x_1(t)(\omega + a_1x_1(t) + a_2x_2(t)) + d_2x_2(t) + F_2)\,dt + \frac{\varepsilon}{\gamma_2}(L_{24}F_4 - L_{24}L_{24}x_2(t))dt \tag{2.14b}$$

$$\frac{\sigma^2}{2} + \varepsilon\frac{1}{\gamma}(L_{13} + b\,x(t))\,b\,dt_{123\,2}$$

$$+ \sqrt{\varepsilon}\frac{\sigma_2}{\gamma_2}L_{24}\,dW_2(t) \tag{2.14c}$$

Note that coarse graining time as t ^{t} amounts to setting $\varepsilon = 1$.

To evaluate the performance of the reduced dynamics we calculate the autocorrelation function $\rho(s) = \frac{<x'(t+s)x'(t)>}{<x'(t)x'>}$ and the third-order two-time moment $K(s) = \frac{<x'^2(t+s)x'(t)>}{<x(t)'^2>^{\frac{3}{2}}}$, which is a measure of deviations from Gaussianity (MTV 2002; Majda

cellent agreement for moderate values of $\varepsilon = 0.1$, 0.5 and still good agreement for $\varepsilon = 1.0$ *et al.* 2005). The comparison of the reduced model with the full model results shows ex- for $\varepsilon = 0.1$ and also $\varepsilon = 0.5$, even though slightly less well. For large values of ε the (Figs. 1 and 2). Especially, the non-Gaussian

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features are reproduced with high accuracy

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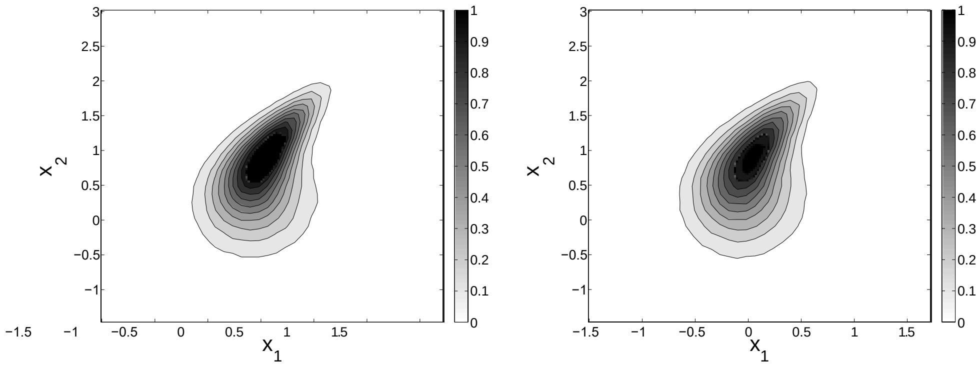

Figure 2. Joint PDF of the slow modes x_1 and x_2 for $\varepsilon = 0.1$. Left column: stochastic dynamics, Right column: reduced dynamics.

reduced dynamics get the sign of the non-Gaussianity right. All of the reduced models in (2.11), (2.12), (2.13) and (2.14) cannot be approximated by the interesting regression strategy of Kravtsov *et al.* 2005) because there is nonlinear multiplicative noise in (2.13) and (2.14), nonlinear triad interaction in (2.13) and augmented cubic nonlinearity in (2.14). Thus these regression strategies necessarily have large model error in this example.

(*f*) *A Mathematical Framework for the Mean Tendency Equation as a Dynamical Diagnostic*

In this section we provide a general framework to estimate the origin of nonlinear signatures of planetary wave dynamics and how important the observed deviations from Gaussianity are for the planetary waves (Franzke *et al.* (2007). This general framework diagnoses contributions to the mean values of state dependent tendencies in a low dimen- sional subspace of complex geophysical systems. The mean phase space tendencies in some GCMs show distinct nonlinear signatures in certain planes which are spanned by its leading EOFs (Branstator & Berner 2005; Selten & Branstator 2004; Franzke *et al.* 2007). Those leading EOFs also show only weak deviations from Gaussianity; mostly in the form of weak skewness and kurtosis and in the case of joint PDFs multiple radial ridges of en- hanced density (Berner & Branstator 2007; Franzke & Majda 2006). This mathematical framework can be applied to complex geophysical systems and reveals how much the self- interaction amongst the modes spanning those planes (climate modes) and how much the unresolved modes contribute to these mean phase space tendencies and also the effect of the observed small deviations from Gaussianity.

To derive the mean tendency equation for the resolved modes we split the state vector of a quadratic dynamical system into resolved modes α and unresolved modes β and utilise conditional mean probability density relations (see Franzke *et al.* 2007 for more details); there we derived a general formula for the *conditional mean tendencies* for the dynamics

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in the resolved variables.

$$< \frac{\partial \alpha_i}{\partial t} | \alpha > = F_i + \sum_{j\in I_R} L_{ij}\alpha_j + \sum_{j,k\in I_R} B_{ijk}\alpha_j\alpha_k \qquad (2.15a)$$

$$+ \sum_{j\in I_U} L_{ij} < \beta_j | \alpha > \qquad (2.15b)$$

$$+2 \sum_{j\in I_R; k\in I_U} B_{ijk}\alpha_j < \beta_k | \alpha > \qquad (2.15c)$$

$$+ \sum_{j,k\in I_U} B_{ijk} < \beta_j\beta_k | \alpha > \qquad (2.15d)$$

Note that the right hand side of (2.15a) is the *bare truncation* restricted to interactions among the resolved modes, while (2.15b) and (2.15c) involve all of the conditional mean statistics $< \beta_j \; \alpha >$; the terms of (2.15c) are associated with *multiplicative triad interac- tions* (leading to multiplicative noise in a MTV framework (MTV 1999; 2003)) of β_j; the terms in (2.15d) involve all of the conditional interaction statistics $< \beta_j\beta_k \; \alpha >$ of second moments and include all of the *additive triad interactions* (leading to additive noise in a MTV framework) of β_j and β_k. Thus, Eq. 2.15 provides a general framework to investigate the mean phase space tendencies which allows the decomposition into contributions from interactions among the resolved planetary waves themselves and various contributions in- volving unresolved degrees of freedom.

Now we simplify the above conditional mean tendency equation for purely Gaus- sian EOF modes. The PDFs in GCMs are nearly Gaussian (Hsu & Zwiers 2001; Berner & Branstator 2007; Franzke *et al.* 2005; Franzke & Majda 2006; Majda *et al.* 2006a), and there are many geophysical models without damping and forcing that exactly satisfy these assumptions such as barotropic flow on the sphere with topography (Salmon 1998; Carnevale & Frederiksen 1987; Majda & Wang 2006); thus, it is reasonable as a starting point to *assume* that the PDF is *exactly Gaussian*. Thus, in the EOF basis the PDF factors like

$$p(\alpha,\beta) = \left(\prod_i p^G(\alpha_i)\right)\left(\prod_j p^G(\beta_j)\right) \qquad (2.16)$$

where $p_i^G(\alpha_i)$, $p_j^G(\beta_j)$ are Gaussian distributions with mean zero. Thus, in the *Gaussian* case, the conditional mean tendency equation simplifies to

$$< \frac{\partial \alpha_i}{\partial t} | \alpha > = F_i + \sum_j L_{ij}\alpha_j + \sum_{j,k} B_{ijk}\alpha_j\alpha_k \qquad (2.17a)$$

$$+ \sum_j B_{ijj}\lambda_j \qquad (2.17b)$$

Note that contributions from (2.15b) and (2.15c), i.e. linear coupling and multiplicative triad interactions, are identically zero. Thus, in this Gaussian case the conditional mean tendency equation recognises bare truncation (2.15a) and a constant forcing from addi- tive triad interactions (2.15d). Since the GCM's have PDF's with only small departures from Gaussianity the behaviour in (2.16) and (2.17) serve as a 'null hypothesis' for these deviations from Gaussianity in climate models.

This general framework predicts that in the case of purely Gaussian modes the non- linear signatures are stemming from the bare truncation (i.e. the self-interaction amongst

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the planetary waves resolved in the low-dimensional plane). In Franzke *et al.* (2007) these diagnostics were applied to a plane of two low-frequency EOF's in a three layer climate model with a nonlinear double swirl; the origin of this double swirl is primarily the three contributions in (2.15b, c and d) from the unresolved modes and not nonlinear effects from the bare truncation.

3. Coarse Grained Stochastic Lattice Models for Climate: Tropical Convection

The current practical models for prediction of both weather and climate involve general cir- culation models (GCM) where the physical equations for these extremely complex flows are discretized in space and time and the effects of unresolved processes are parametrised according to various recipes. With the current generation of supercomputers, the small- est possible mesh spacings are about 10-50 km for short-term weather simulations and of order 100 km for short term climate simulations. There are many important physical processes which are unresolved in such simulations such as the mesoscale sea-ice cover, the cloud cover in sub-tropical boundary layers, and deep convective clouds in the trop- ics. Most of these features are highly intermittent in space and time. An appealing way to represent these unresolved features is through a suitable coarse-grained stochastic model which simultaneously retains crucial physical features of the interaction between the unre- solved and resolved scales in a GCM. In work from 2002 and 2003, two of the authors both have developed a new systematic stochastic strategy (Majda & Khouider 2002, Khouider *et al.* 2003) to parametrise key features of deep convection in the tropics involving suitable stochastic spin-flip models and also a systematic mathematical strategy to coarse-grain such microscopic stochastic models to practical mesoscopic meshes in a computationally efficient manner while retaining crucial physical properties of the interaction.

As regards tropical convection, crucial scientific issues involve the fashion in which a stochastic model effects the climate mean state and the strength and nature of fluctua- tions about the climate mean. Here the strategy to develop a new family of coarse-grained stochastic models for tropical deep convection is briefly reviewed (Majda & Khouider 2002, Khouider *et al.* 2003) as an illustrative example of the potential use of stochastic lattice models. In (Khouider *et al.* 2003), it has been established that in suitable regimes of parameters, the coarse grained stochastic parametrisations can significantly alter the climatology as well as increase wave fluctuations about the climatology. This was estab- lished in (Khouider *et al.* 2003) in the simplest scenario for tropical climate involving the Walker circulation, the east-west climatological state which arises from local region of enhanced surface heat flux, mimicking the Indonesian marine continent. Convectively coupled waves in the tropics such as the Madden-Julian oscillation play an important role in medium range forecasts yet the current generation of computer models fail to represent such waves adequately (Lin *et al.* 2006). Palmer (2001) has emphasised the potential of stochastic parametrisation to reduce the model error in a deterministic computer model. Here in an idealised setting, we show how to develop a stochastic parametrisation to mod- ify and improve the behaviour of convectively coupled waves in a reasonable prototype GCM; this is achieved by following a path guided by the systematic design principle for the idealised model in (1.1) (KMS 2006, 2007) to build in suitable intermittency effects.

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(a) The Microscopic Stochastic Model for CIN

the generic vector, $\dot{u}$, are regarded as known only over a discrete horizontal mesh with i_j In a typical GCM, the fluid dynamical and thermodynamical variables, denoted here by $\dot{u}(j\Delta x, t)$ denoting these discrete values. Throughout the discussion, one horizontal spatial dimension along the equator in the east-west direction is assumed for simplicity in notation and explanation. As mentioned above, the typical mesh spacing in a GCM is coarse with

Δx ranging from 50 km to 250 km depending on the time duration of the simulation. The stochastic variable used to illustrate the approach is convective inhibition. Observationally,
convective inhibition (CIN) is known to have significant fluctuations on a horizontal spatial scale on the order of a kilometer, the microscopic scale here, with changes in CIN attributed to different mechanisms in the turbulent boundary layer such as gust fronts, gravity waves, and turbulent fluctuations in equivalent potential temperature (Mapes 2000). In (Khouider *et al.* 2003), it was proposed that all of these different microscopic physical mechanisms and their interaction which increase and decrease CIN are too complex to model in detail in a coarse mesh GCM parametrisation and instead, as in statistical mechanics, should be modelled by a simple order parameter, σ_I, taking only two discrete values,

$$\begin{aligned} \sigma_I &= 1 && \text{at a site if convection is inhibited (a CIN site)} \\ \sigma_I &= 0 && \text{at a site if there is potential for deep convection (a PAC site).} \end{aligned} \qquad (3.1)$$

The value of CIN at a given coarse mesh point is determined by the averaging of CIN over the microscopic states in the vicinity of the given mesh point, i.e.,

$$\bar{\sigma}_I(j\Delta x, t) = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \sigma_I(x, t)\, dx. \qquad (3.2)$$

Note that the mesh size, Δx, is mesoscopic, i.e., between the microscale, $O(1\text{ km})$, and the macroscale, $O(10,000\text{ km})$, and that $\bar{\sigma}_I$ can have any value in the range $0 \leq \bar{\sigma}_I \leq 1$. Discrete sums over microscopic mesh values (of order 1 km) and continuous integrals are utilised interchangeably for notational convenience (Majda & Khouider 2002).

(b) The Simplest Coarse-Grained Stochastic Model

In practical parametrisation, it is desirable for computational feasibility to replace the microscopic dynamics by a process on the coarse mesh which retains critical dynamical features of the interaction. Following the general procedure developed and tested in (Kat- soulakis *et al.* 2003a, Katsoulakis *et al.* 2003b, Katsoulakis & Vlachos 2003) the simplest local version of the systematic coarse grained stochastic process is developed in (Khouider *et al.* 2003) and summarised here.

Each coarse cell Δx_k, $k = 1, \ldots, m$, of the coarse-grained lattice is divided onto q microscopic cells such that $\Delta x_k^1 \{1, 2, \ldots, q\}$, $k = 1, \ldots, m$. In the coarse-grained procedure, given the coarse-grained sequence of random variables

$$\eta_t(k) = \sum_{y \in \Delta x_k} \sigma_{I,t}(y), \qquad (3.3)$$

so that the average in (3.2) verifies $\bar{\sigma}_I(j\Delta x) = \eta(k)/q$, for $j = k$ in some sense, the microscopic dynamics is replaced by a birth/death Markov process defined on the variables,

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$\{0, 1, \cdots, q\}$, for each k such that $\eta_t(k)$ evolves according to the following probability law.

$$\text{Prob}\left\{\eta_{t+\Delta t}(k) = n+1 \mid \eta_t(k) = n\right\} = C_a(k,n)\Delta t + o(\Delta t)$$
$$\text{Prob}\left\{\eta_{t+\Delta t}(k) = n-1 \mid \eta_t(k) = n\right\} = C_d(k,n)\Delta t + o(\Delta t) \tag{3.4}$$
$$\text{Prob}\left\{\eta_{t+\Delta t}(k) = n \mid \eta_t(k) = n\right\} = 1 - \left[C_a(k,n) + C_d(k,n)\right]\Delta t + o(\Delta t)\ \text{Prob}\left\{\eta_{t+\Delta t}(k) \neq n, n-1, n+1 \mid \eta_t(k) = n\right\} = o(\Delta t).$$

The coarse grained adsorption/desorption rates are given respectively by

$$C_a(k,\eta) = \frac{1}{\tau_I}[q - \eta(k)]$$
$$C_d(k,\eta) = \frac{1}{\tau_I}\eta(k)e^{-\beta\bar{V}(k)} \tag{3.5}$$

where

$$\bar{V}(k) = \bar{J}(0,0)\left(\eta(k) - 1\right) + h_{\text{ext}} \tag{3.6}$$

with the coarse grained interaction potential within the coarse cell given by $\bar{J}(0,0) = 2U_0/(q-1)$ where U_0 is the mean strength of the potential J (Katsoulakis *et al.* 2003a, Kat- soulakis *et al.* 2003b). The coarse-grained energy content for CIN is given by the coarse-grained Hamiltonian

$$\bar{H}(\eta) = \frac{U_0}{q-1}\sum_k \eta(k)\left(\eta(k)-1\right) + h_{\text{ext}}\sum_k \eta(k). \tag{3.7}$$

The canonical invariant Gibbs measure for the coarse-grained stochastic process is a prod- uct measure given by

$$G_{m,q,\beta}(\eta) = (Z_{m,q,\beta})^{-1}e^{\beta\bar{H}(\eta)}P_{m,q}(d\eta) \tag{3.8}$$

where $P_{m,q}(d\eta)$ is an explicit prior distribution (Katsoulakis *et al.* 2003b). As shown in (Katsoulakis *et al.* 2003b), the coarse-grained birth/death process above satisfies detailed balance with respect to the Gibbs measure in (3.8) as well as a number of other attractive theoretical features. The simplest coarse-grained approximation given above assumes that the effect of the microscopic interactions on the mesoscopic scales occurs within the meso- scopic coarse-mesh scale, Δx, otherwise systematic nonlocal couplings are needed (Kat- soulakis *et al.* 2003b). The accuracy of these approximations is tested for diverse examples from material science elsewhere (Katsoulakis *et al.* 2003a, Katsoulakis *et al.* 2003b, Kat- soulakis & Vlachos 2003) and for the instructive idealised coupled models in (1.1) (KMS, 2004,2005,2006, 2007).

requires specification of the parameters, τ_I, U_0, q and the external potential $h_{\text{ext}}(\dot{u}_j)$. The practical implementation of the coarse-grained birth/death process in (3.3)–(3.6) well as the statistical parameter β. The advantages of such a stochastic lattice model are the following:

1) Retains systematically the energetics of unresolved features through the coarse- grained Gibb's measure

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2) Has minimal computational overhead since there are rapid algorithms for updating birth death processes

3) Incorporates feedbacks of the resolved modes on the unresolved modes and there are energetics through an external field

4) Includes dynamical coupling through not only sampling the probability distributions of unresolved variables but also their evolving behaviour in time is constrained by the large scale dynamics.

(c) The Model Deterministic Convective Parametrisation

A prototype mass flux parametrisation with crude vertical resolution (Majda & Shefter 2001, Khouider & Majda 2006b) is utilised to illustrate the fashion in which the coarse- grained stochastic model for CIN can be coupled to a deterministic convective mass flux

parametrisation. The prognostic variables (u, θ, θ_{eb}, θ_{em}) are the x-component of the fluid velocity, u, the potential temperature in the middle troposphere, θ, the equivalent potential temperatures, θ_{eb} and θ_{em}, measuring, respectively, the potential temperatures plus mois- ture content of the boundary layer and middle troposphere. The vertical structure is deter- mined by projection on a first baroclinic heating mode (Majda & Shefter 2001; Khouider & Majda, 2006b). The dynamic equations for these variables in the parametrisation are given by

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \bar{\alpha}\frac{\partial \theta}{\partial x} &= -C_D^0 \frac{1}{h}\left(u^2 + u_0^2\right)^{1/2} u - \frac{1}{\tau_D} u \\
\frac{\partial \theta}{\partial t} - \bar{\alpha}\frac{\partial u}{\partial x} &= S - Q_R^0 - \frac{\theta}{\tau_R} \\
h\frac{\partial \theta_{eb}}{\partial t} &= -D(\theta_{eb} - \theta_{em}) + C_\theta^0 \left(u^2 + u_0^2\right)^{1/2} (\theta_{eb}^* - \theta_{eb}) \\
H\frac{\partial \theta_{em}}{\partial t} &= D(\theta_{eb} - \theta_{em}) - HQ_R^0 - H\frac{\theta_{em}}{\tau_R}
\end{aligned} \tag{3.9}
$$

while the constants Q_R^0, θ_{eb}^* are externally imposed and represent the radiative cooling at equilibrium in the upper troposphere and saturation equivalent potential temperature in the boundary layer. The constants h and H measure the depths of the boundary layer and the troposphere above the boundary layer, respectively. The typical values used here are

h = 500 m and H = 16 km while u_0 = 2 m s^{-1}. The explicit values for the other constants used in (3.9) and elsewhere in this section can be found in (Majda & Shefter 2001, Khouider *et al.* 2003).

The crucial quantities in the prototype mass flux parametrisation are the terms S and D where S represents the middle troposphere heating due to deep convection while D represents the downward mass flux on the boundary layer. The heating term S is given by

$$
S = M\sigma_c \left(\mathrm{CAPE}\right)_+^{1/2} \tag{3.10}
$$

with M a fixed constant, σ_c the area fraction for deep convective mass flux, and CAPE = $R(\theta_{eb}\ \gamma\theta)$, the convectively available potential energy. Here R is a dimensional constant (Majda & Shefter, 2001, Khouider *et al.* 2003). The downward mass flux on the boundary

layer, D, includes the environmental downdrafts, m_e, and the downward mass flux due elsewhere (Majda & Shefter, 2001; Khouider *et al.* 2003). The notation $(X)^+$ denotes the to convection, m_-, which are non-negative quantities with explicit formulas described positive part of X. This parametrisation respects conservation of vertically integrated moist

static energy.

The equations in (3.9), (3.10) represent an idealised GCM with crude vertical resolu- tion based on reasonable design principle for deep convection including basic conservation principles. However, like the current generation of GCM's, the model has major deficien- cies as regards convectively coupled waves. First, instabilities for the full model in (3.9)

jda & Shefter, 2001). There is no observational evidence supporting the role of WISHE in driving the large scale convectively coupled waves in nature (Lin *et al.* 2006 and references therein); here the WISHE

arise only through the nonlinearity from $u + u$ and are called the WISHE

term is regarded as a deterministic fix in a GCM parametrisation to generate instabilities. The simulation results reported below in Figure 5a show that in an aquaplanet model above the equator, two regular periodic convectively coupled waves, moving eastward and westward at roughly equal strength are generated by the idealised GCM. Since GCM's often have convectively coupled waves that move too fast and are too regular (Lin *et al.* 2006), the goal here is to see whether the stochastic lattice coupling will slow down the waves in the deterministic parametrisation and simultaneously increase the spatio-temporal fluctuations in those waves. It is important to note here that there are recent deterministic multi-cloud models (Khouider & Majda, 2006a, 2007a, 2007b) for convec- tively coupled waves involving the three cloud types in observations above the boundary layer, congestus, deep, and stratiform, and their heating structure which reproduce key fea- tures of the observational record for convectively coupled waves (Lin *et al.* 2006). The mechanism of instability in these models (Khouider & Majda, 2006a) is completely differ- ent from WISHE which is not active in the multicloud models. For the idealised setting of flow above the equator, the multicloud models can produce packets of convectively coupled waves moving in one direction at 15-20 m/s with their low frequency envelopes moving at 4-7 m/s in the opposite direction across the warm pool in a fashion like the Madden-Julian oscillation (Khouider & Majda, 2007a, b).

(d) Coupling of the Stochastic CIN Model into the Parametrisation

The equations in (3.9)–(3.10) are regarded here as the prototype deterministic GCM parametrisation when discretized in a standard fashion utilising central differences on

a coarse mesh Δx with Δx ranging from 50 km to 250 km. In the simulations from (Khouider *et al.* 2003), and presented below Δx = 80 km. The coarse-grained stochas- tic CIN model is coupled to this basic parametrisation. First, the area fraction for deep

convection, σ_c, governing the upward mass flux strength, is allowed to vary on the coarse mesh and is given by

$$\text{with } \bar{\sigma}_I \text{ is the average in (3.2)} \quad \sum \quad \sum \quad \sigma = \sigma_c^+ (1 - \bar{\sigma}) \qquad (3.11)$$

with σ_c^+ a threshold constant, σ_c^+ = .002 (Majda & Shefter 2001, Khouider *et al.* 2003).

When the order parameter σ_I signifies strong CIN locally so that $\bar{\sigma}_I$ = 1, the flux of deep convection is diminished to zero while with PAC locally active, $\bar{\sigma}_I$ = 0, this flux increases

CIN model into the parametrisation, the coarse mesh external potential, $h_{\text{ext}}(\hat{u}_j)$, from to the maximum allowed by the value σ^+. To complete the coupling of the stochastic (3.6), (3.7), needs to

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be specified from the coarse mesh values, $\dot{u}_j$. There is no unique
choice of the external potential but its form can be dictated by simple physical reasoning. In (Khouider *et al.* 2003), the plausible physical assumption is made that when the convective downward mass flux, m_-, decreases, the energy for CIN decreases. Since the convective downward mass flux results from the evaporative cooling induced by precipitation falling into dry air, it constitutes a mechanisms which carries negatively buoyant cool and dry air from the middle troposphere onto the boundary layer hence tending to reduce CAPE and deep convection.

Another natural external potential is the boundary layer equivalent potential tempera- ture since the flux at the boundary is crucial physically. Thus, the choice

$$h_{ext}(j\Delta x, t) = \tilde{\gamma}\theta_{eb}(j\Delta x, t) \quad (3.12)$$

is utilised here with $\tilde{\gamma}$ a calibration factor. The other parameters in the stochastic lattice model are chosen as τ_I = 2 hours, $\beta = 1$, $U_0 = 1$ so that CIN sites are favoured in the equilibrium Gibbs measure.

(e) *The stochastic single column model and intermittency*

A central issue is how to calibrate the stochastic lattice model to generate intermittent fluctuations with plausible magnitudes as observed in tropical convection. A natural design framework is first to achieve such behaviour in the stochastic single column model given by the equations

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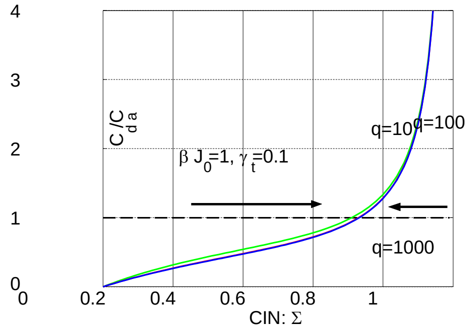

Figure 3. Ratio of the adsorbtion and desorption rates C_d/C_a for the birth-death stochastic lattice model with parameters $\tilde{\gamma} = 0.1, \beta = U_0 = 1$ and a number of microscopic sites $q = 10$ (green), $q = 100$ (red), and $q = 1000$ (blue). Note a CIN dominating equilibrium with large CIN values yield large PAC (adsorbtion) rates. The balanced equilibrium curve $C_d/C_a = 1$ is shown.

$q = 10$. The strongly intermittent fluctuations in θ_{eb} over several degrees Kelvin as well as similar intermittency in the mass flux is evident. Also note that the fluctuations in the mid-
troposphere potential temperature are much weaker in magnitude as occurs in the actual tropics.

(f) The effects of the stochastic parametrisation on convectively coupled waves

All parameters in the stochastic lattice model have been determined through system- atic design principles in the previous section. Here the results of numerical simulations of the stochastic model in statistical steady state are reported for flow above the equator in a standard aquaplanet set-up with uniform SST (Majda & Khouider 2002; Khouider *et al.* 2003). The results of various simulations are reported in Figure 5 for comparison with the deterministic parametrisation with WISHE reported in Figure 5A. As shown in Figure 5B, the effect of the fluctuations of the stochastic lattice model is simultaneously to cre- ate more realistic intermittency in the convectively coupled waves and to slow down their phase speed from 15 m/s to 11 m/s; as mentioned earlier, these are desirable qualitative features of a stochastic parametrisation. In Figure 5C, we report the result of running the model parametrisation in (3.9) without WISHE but coupled to the stochastic lattice model; recall that this is the situation where the deterministic model without WISHE produces no waves in the statistical steady state. In Figure 5C, there is a clear evidence for stochas- tic generated convectively coupled waves with the reduced phase speed of roughly 8 m/s and roughly half the amplitude; these waves have been created by coupling alone to the intermittent stochastic lattice model and without deterministic instability. This is another attractive feature of the present stochastic lattice models in changing the character of model error for convection (Palmer, 2001).

4. Concluding Remarks

This paper both reviewed and provided new illustrations and examples of the fashion in which modern applied mathematics can provide new perspectives and systematic de- sign principles for stochastic modelling for climate. Section 2 was devoted to systematic stochastic modelling of low-frequency variability including the quantitative sources for

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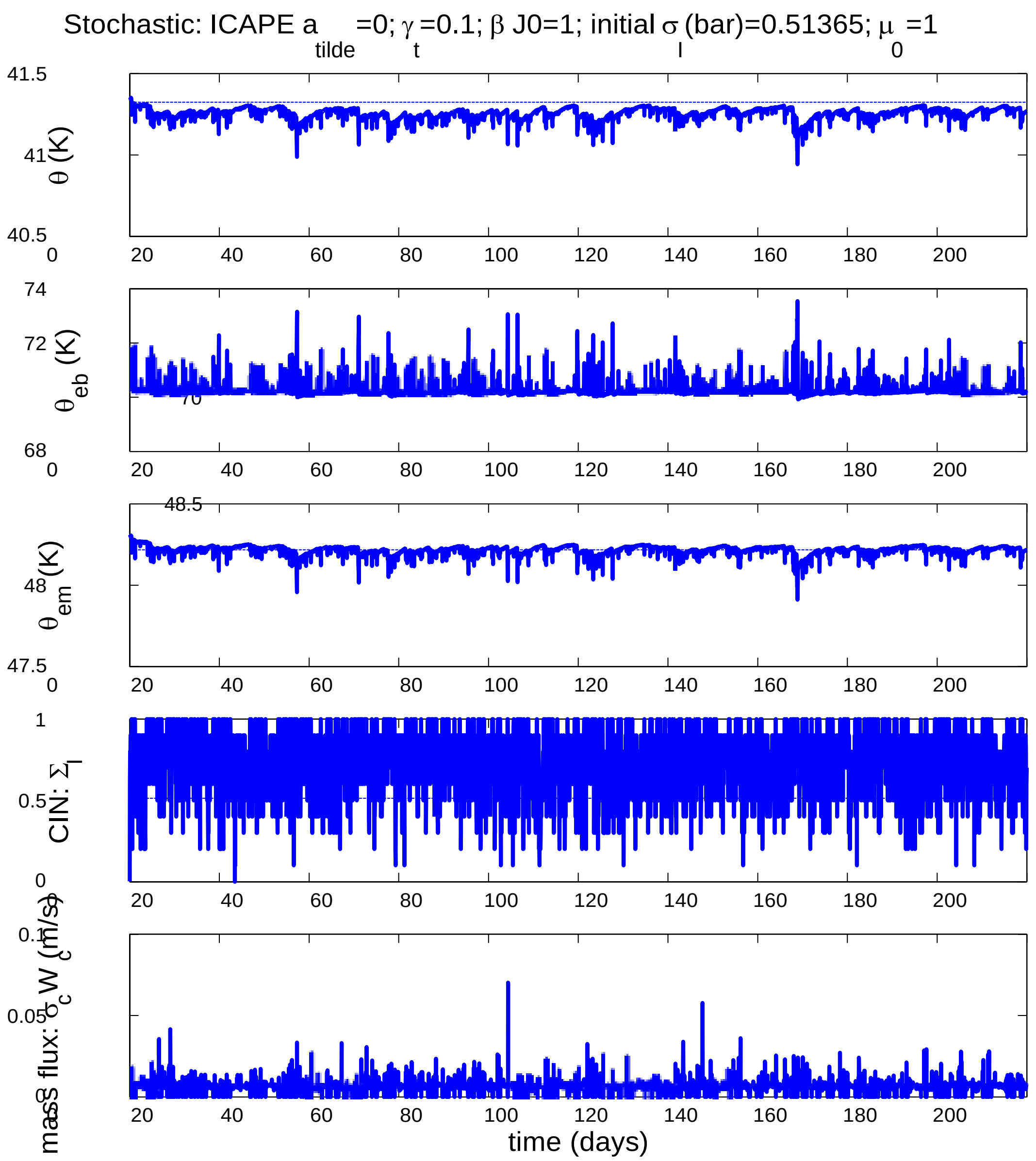

Figure 4. Time series of dynamical variables for the stochastic one column model with $\tau_I = 2\,\text{hours}, \tilde{\gamma} = 0.1, \beta = U_0 = 1, q = 10$. Note the manifestation of strong intermittent bursts beyond the stochastically generated CIN; especially, in θ_{eb} and mass flux $\sigma_c W_c\,(1 \;\; \sigma_I)\; R(\theta_{eb} \;\; \gamma\theta_1)$. The deterministic equilibrium value is represented by the hori- zontal line on each panel.

multiplicative noise. A new simplified low-dimensional stochastic model with key features of atmospheric low-frequency variability was introduced in 2e in order to test stochastic mode reduction strategies. A recent diagnostic test with firm mathematical underpinning for exploring the subtle departures from Gaussianity and their sources was discussed in

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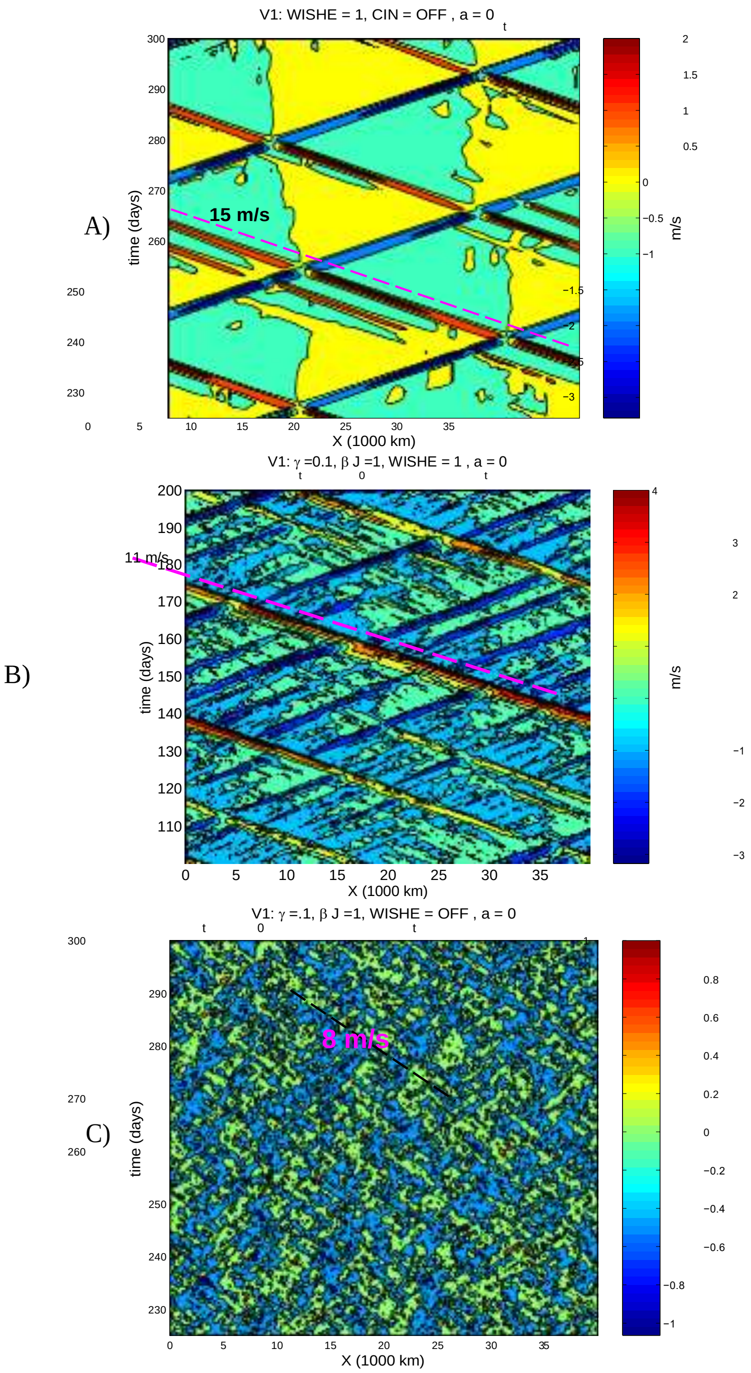

Figure 5. Effect of the birth-death stochastic CIN on large scale convectively coupled waves with the stochastic parameters $\beta = U_0 = 1$, $q = 10$. (A) Purely deterministic WISHE waves without the stochastic coupling, (B) Effect of stochastic CIN on WISHE waves, (C) Intermittent bursts of convec-
tive episodes with apparent tracks of waves maintained by the stochastic-CIN effects alone–WISHE is off. The apparent speed of propagation is shown by the dashed line on each panel.

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2f. Section 3 was devoted to developing stochastic lattice models to capture intermittent features and improve the fidelity of deterministic parametrisations. Recent, systematic de- sign principles (KMS 2006, 2007) were utilised to calibrate a stochastic column model for tropical convection in 3e; the practical effect of slowing down convectively coupled waves and increasing their fluctuations through these stochastic lattice models were presented in 3f.

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References

Achatz, U. & Branstator G. 1999 A two-layer model with empirical linear corrections and reduced order for studies of internal climate variability. *J. Atmos. Sci.* **56**, 3140–3160.

Berner, J. 2005 Linking Nonlinearity and non-Gaussianity of Planetary Wave Behaviour by the Fokker-Planck equation. *J. Atmos. Sci.* **62**, 2098–2117.

Berner, J. & Branstator, G. 2007 Linear and Nonlinear Signatures in the Planetary Wave Dynamics of an AGCM: Probability Density Functions. *J. Atmos. Sci.*, **64**, 117–136.

Branstator, G. 1990 Low-Frequency Patterns Induced by Stationary Waves. *J. Atmos. Sci.* **47**, 629– 648.

Branstator, G. & Berner, J. 2005 Linear and Nonlinear Signatures in the Planetary Wave Dynamics of an AGCM: Phase Space Tendencies. *J. Atmos. Sci.*, **62**, 1792-1811.

Branstator, G. & Haupt S. E. 1998 An empirical model of barotropic atmospheric dynamics and its response to tropical forcing. *J. Climate* **11**, 2645–2667.

Carnevale, G. & Frederiksen, J. S. 1987 Nonlinear stability and statistical mechanics for flow over topography. *J. Fluid Mech.*, **175** , 157–181.

Crommelin, D. T. & Majda, A. J. 2004 Strategies for model reduction: comparing different optimal bases. *J. Atmos. Sci.* **61**, 2206–2217.

Crommelin, D. T. & Vanden-Eijnden, E. 2006 Fitting Timeseries by Continuous-Time Markov Chains: A Quadratic Programming Approach. *J. Comp. Phys.* **217**, 782–805.

Delsole, T. 2004 Stochastic models of quasi-geostrophic turbulence. *Surveys Geophys.* **25**, 107-149. Egger, J. 2005
A stochastic model for the angular momentum budget of latitude belts. *J. Atmos. Sci.* **62**, 2592–2601.

Feldstein, S. B. 2000 The timescale, power spectra, and climate noise properties of teleconnection patterns. *J. Climate* **13**, 4430–4440.

Franzke, C. & Majda, A. J. 2006 Low-order stochastic mode reduction for a prototype atmospheric GCM. *J. Atmos. Sci.* **63**, 457–479.

Franzke, C., Majda, A. J. & Branstator, G. 2007 The Origin of Nonlinear Signatures of Planetary Wave Dynamics: Mean Phase Space Tendencies and Contributions from Non-Gaussianity. *J. At- mos. Sci.*, **64**, in press.

Franzke, C., Majda, A. J. & Vanden-Eijnden, E. 2005 Low-order stochastic mode reduction for a realistic barotropic model climate.*J. Atmos. Sci.* **62**, 1722–1745.

Frederiksen, J. S., Dix, M. R. & Kepert, S. M., 1996 Systematic energy errors and the tendency toward canonical equilibrium in atmospheric circulation models. *J. Atmos. Sci.*, **53**, 887–904.

Gardiner, C. W. 1985 *Handbook of Stochastic Methods* Springer-Verlag, 442 pp.

Hasselmann, K. 1988 PIPs and POPs: The reduction of complex dynamical systems using principal interaction and oscillation patterns. *J. Geophys. Res.* **93**, 11015–11021.

Hsu, C. J. & Zwiers, F. 2001 Climate Change in Recurrent Regimes and Modes of Northern Hemi- sphere Atmospheric Variability. *J. Geophys. Res.*, **106**, 20145–20159.

Hurrell, J. W. 1995 Decadal Trends in the North Atlantic Oscillation: Regional Temperatures and Precipitation. *Science* **269**, 676–679.

Katsoulakis, M. A., Majda, A. J. & Sopasakis, A. 2004 Multiscale coupling in prototype hybrid deterministic/stochastic systems. Part I: Deterministic closures. *Comm. Math. Sci.* **2**, 255-294.

Katsoulakis, M. A., Majda, A. J. & Sopasakis, A. 2005 Multiscale coupling in prototype hybrid deterministic/stochastic systems. Part II: Stochastic closures. *Comm. Math. Sci.* **3**, 453–478.

Katsoulakis, M. A., Majda, A. J. & Sopasakis, A. 2006, Intermittency, metastability, and coarse graining for coupled deterministic stochastic lattice systems. *Nonlinearity.* **19**, 1021–1047.

Katsoulakis, M. A., Majda, A. J. & Sopasakis, A. 2007, Prototype hybrid couplings of macroscopic deterministic models and microscopic stochastic lattice dynamics. *Contemporary Mathematics.* **429**, 143–187.

ISSN: 2278-4632
Vol-12 Issue-07 No.03 July 2022

Katsoulakis, M. A., Majda, A. J. & Vlachos, G. D. 2003a Coarse-grained stochastic processes for microscopic lattice systems. *Proc. Nat. Acad. Sci. USA*, **100**, 782-787.

Katsoulakis, M. A., Majda, A. J. & Vlachos, G. D. 2003b Coarse-grained stochastic processes and Monte Carlo simulations in lattice systems. *J. Comp. Phys.* **186**, 250–278.

Katsoulakis, M. A. & Vlachos, G. D. 2003 Hierarchical kinetic Monte Carlo simulations for diffusion of interacting molecules. *J. Chem. Phys.* **112**, 9412–9427.

Khouider, B. & Majda, A. J. 2006a A simple multicloud parametrization for convectively coupled tropical waves. Part I: Linear analysis. *J. Atmos. Sci* **63**, 1308–1323.

Khouider, B. & Majda, A. J. 2006b Model multicloud parametrizations with crude vertical structure *Theoretical and Computational Fluid Dynamics* **20**–Special Issue: Theoretical developments in tropical meteorology, 351–375.

Khouider, B. & Majda, A. J. 2007a A simple multicloud parametrization for convectively coupled tropical waves. Part II: Nonlinear simulations *J. Atmos. Sci.* **64**, 381–400.

Khouider, B. & Majda, A. J. 2007b Multicloud models for tropical convection: enhanced congestus heating *J. Atmos. Sci.* (In press).

Khouider, B., Majda, A. J. & Katsoulakis, M. 2003, Coarse grained stochastic models for tropical convection. *Proc. Nat. Acad. Sci. USA* **100**. 11941–11946.

Kravtsov, S., Kondrashov D. & Ghil, M. 2005 Multi-level regression modeling of nonlinear pro- cesses: Derivation and applications to climate variability. *J. Climate* **62**, 4404–4424.

Kwasniok, F. 1996 The reduction of complex dynamical systems using principal interaction patterns. *Physica D*, **92**, 28–60.

Kwasniok, F. 2004 Empirical low-order models of barotropic flow. *J. Atmos. Sci.* **61**, 235–245.

Kurtz, T. G., 1973 A limit theorem for perturbed operators semigroups with applications to random evolution. *J. Funct. Anal.*, **12**, 55–67.

Lin, J-L, G. N. Kiladis, B. E. Mapes, K. M. Weickmann, K. R. Sperber, W. Lin, M. Wheeler, S. D. Roeckner & J. F. Scinocca, 2006 Tropical intraseasonal variability in 14 IPCC AR4 climate models. Part 1: Convective signals. *J. Climate.* **19**, 2665–2690.

Majda, A. J. 2000 Real world turbulence and modern applied mathematics. *Mathematics Frontiers and Perspectives*. **2000**, Amer. Math. Soc., 137–151.

Majda, A. J., Abramov, R. & Grote, M. 2005 *Information Theory and Stochastics for Multiscale Non- linear Systems*. Monograph Series of Center for Research in Mathematics, University of Montreal, Published by American Mathematical Society, 140p.

Majda, A. J., Franzke, C., Fischer, A. & Crommelin, D. T. 2006a Distinct Metastable Atmospheric Regimes Despite Nearly Gaussian Statistics: A Paradigm Model. *Proc. Natl. Acad. Sci. USA*, **103**, 8309–8314.

Majda, A. J. & Khouider, B. 2002 Stochastic and mesoscopic models for tropical convection. *Proc. Nat. Acad. Sci. USA*, **99**, 1123-1128.

Majda, A. J. & Shefter, M. 2001 Waves and instabilities for model tropical convective parametriza- tions. *J. Atmos. Sci.* **58**, 896-914.

Majda, A. J. & Timofeyev, I. 2000 Remarkable statistical behavior for truncated Burgers-Hopf dy- namics. *Proc. Natl. Acad. Sci. USA* **97**, 12413–12417.

Majda, A. J. & Timofeyev, I. 2004 Low Dimensional Chaotic Dynamics Versus Intrinsic Stochastic Chaos: A Paradigm Model. *Physica D* **199**, 339-368.

Majda, A. J., Timofeyev, I. & Vanden-Eijnden, E. 1999 Models for stochastic climate prediction. *Proc. Natl. Acad. Sci. USA* **96**, 14687–14691.

Majda, A. J., Timofeyev I. & Vanden-Eijnden, E. 2001 A mathematical framework for stochastic climate models. *Commun. Pure Appl. Math.* **54**, 891–974.

Majda, A. J., Timofeyev, I. & Vanden-Eijnden, E. 2002 A priori tests of a stochastic mode reduction strategy. *Physica D* **170**, 206–252.

Majda, A. J., Timofeyev, I. & Vanden-Eijnden, E. 2003 Systematic strategies for stochastic mode reduction in climate. *J. Atmos. Sci.* **60**, 1705–1722.

ISSN: 2278-4632
Vol-12 Issue-07 No.03 July 2022

Majda, A. J., Timofeyev, I. & Vanden-Eijnden, E. 2006b Stochastic Models for Selected Slow Vari- ables in Large Deterministic Systems. *Nonlinearity* **19**, 769–794.

Majda, A. J. & Wang, X. 2006 *Nonlinear Dynamics and Statistical Theories for Basic Geophysical Flows* Cambridge University Press, 600p.

Mapes, B. E. 2000 Convective inhibition, subgridscale triggering energy, and "stratiform instability" in a toy tropical wave model. *J. Atmos. Sci.* **57**, 1515-1535.

Marshall, J. & Molteni, F. 1993 Toward a Dynamical Understanding of Planetary-Scale Flow Regimes. *J. Atmos. Sci.* **50**, 1792–1818.

Newman, M., Sardeshmukh, P. D. & Penland, C. 1997 Stochastic forcing of the wintertime extrat- ropical flow. *J. Atmos. Sci.* **54**, 435–455.

Palmer, T. N. 2001 A nonlinear dynamical perspective on model error: a proposal for non-local stochastic-dynamic parametrizations in weather and climate prediction models. *Q. J. R. Meteorol. Soc.* **127**, 279–304.

Percival, D. B., Overland, J. E. & Mofjeld, H. O. 2001 Interpretation of North Pacific variability as a short- and long-memory process. *J. Climate* **14**, 4545–4559.

Salmon, R. 1998 *Lectures on Geophysical Fluid Dynamics* Oxford University Press, 378p.

Schubert, S. D. 1985 A statistical-dynamical study of empirically determined modes of atmospheric variability. *J. Atmos. Sci.* **42**, 3–17.

Selten, F. M. 1995 An efficient description of the dynamics of barotropic flow. *J. Atmos. Sci.* **52**, 915–936.

Selten, F. M. & Branstator, G. 2004 Preferred Regime Transition Routes and Evidence for an Unsta- ble Periodic Orbit in a Baroclinic Model. *J. Atmos. Sci.*, **61**, 2267–2282.

Stephenson, D. B., Pavan, V. & Bojariu, R. 2000 Is the North Atlantic Oscillation a random walk? *Int. J. Climatol.* **20**, 1–18.

Sura, P. 2003 Stochastic Analysis of Southern and Pacific Ocean Sea Surface Winds. *J. Atmos. Sci.* **60**, 654–666.

Sura, P. and Sardeshmukh P. D. 2007 A global view of non-Gaussian SST variability. *J. Phys. Ocean.* , in press.

Wallace, J. M. & Gutzler, D. S. 1981 Teleconnections in the geopotential height field during the Northern Hemisphere winter. *Mon. Wea. Rev.* **109**, 784–804.

Whitaker, J. S. & Sardeshmukh, P. D. 1998 A linear theory of extratropical synoptic eddy statistics. *J. Atmos. Sci.* **55**, 237–258.

Zhang, Y. & Held, I. M. 1999 A linear stochastic model of a GCM's midlatitude storm tracks. *J. Atmos. Sci.* **56**, 3416–3435.