Numerical Study of Thermal Stresses in an Annular Disc.

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Abstract- The present paper deals with numerical solution of temperature change and analysis of stresses caused by thermally induced strains that are elastic, on an annular disc. An inverse, steady state thermoelastic problem is solved using finite Marchi- Zgrablich integral transform. Results obtained are numerically verified by considering a special case.

Keywords – Inverse Thermoelastic problem, Marchi –Zgrablich transform.

I. INTRODUCTION.

Thermoelasticity deals with the theory of heat conduction and the theory of strains and thermal stresses, when coupling of temperature and strain fields occurs. Thermoelasticity helps to determine the stresses produced by the temperature field and the distribution of temperature due to the action of internal forces.

The Inverse thermoelastic problem is, knowing the conditions of the displacement and stresses at some points of the solid, aim is to determine the temperature of the heating medium and the heat flux of a solid. The study of thermoelastic behaviour of annular disc is necessary as they constitute the foundations of containers for hard gases or liquids.

Many of researches on thermoelasticity has been performed to find analytical solution of thermoelastic problems. By considering axisymmetric temperature change, W. Nowacki [6] has done the analysis of steady-state thermal stresses in a circular plate . The Heat conduction in hollow cylinder with radiation is discussed in detailed by Marchi E. and Zgrablich G. [1]. A thin clamped circular plate is analysed by K. C. Deshmukh et.al. [3] to study effect of heat generation. Inverse quasi static stresses are studied by I. Khan et.al. [2] considering a transient theromoelastic problem of thick circular plate . Recently, Ahire et. al. [7] elaborated effect of moving heat on thermal stresses considering a thin circular plate.

In this present manuscript, the boundary value problem of uncoupled thermoelasticity of a annular disc is discussed and the temperature distribution, displacement and thermal stresses are obtained. Under the given boundary conditions, the finite Marchi- Zgrablich transform technique is used to find the analytical solution of the inverse, steady state thermoelastic problem.

II. MATHEMATICAL FORMULATION

Consider a homogenous , isotropic thin annular disc of thickness h and occupying the space $D: a \leq r \leq b,\ 0 \leq z \leq h$ and having constant thermal properties. The boundary conditions of radiation type is applied on both curved surfaces $(r = a)$ and $(r = b)$. Heat flux is kept at $F(r)$ on

lower surface $(z = 0)$ whereas on a upper surface $z = h$ of the disc, temperature is kept at $G(r)$, where $G(r)$is unknown function of r.
The differential equation for displacement function $U(r,z)$ as in Nowacki is,

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = (1+\nu)\alpha T \tag{2.1}$$

with $(U_r) = 0$; at $(r = a)$ and $(r = b)$. (2.2)
where,
ν: Poisson's ratio.
α : Linear coefficient of thermal expansion.
The temperature $T(r,z)$ of the disc satisfies the differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{2.3}$$

Subject to boundary conditions

$$\left[h_1 \frac{\partial T(r,z)}{\partial r} + T(r,z)\right]\Big|_{r=a} = f(z) \tag{2.4}$$

$$\left[h_2 \frac{\partial T(r,z)}{\partial r} + T(r,z)\right]\Big|_{r=b} = g(z) \tag{2.5}$$

$$\left[\frac{\partial T(r,z)}{\partial z}\right]\Big|_{z=0} = F(r) \tag{2.6}$$

$$[T(r,z)]|_{z=h} = G(r) \qquad \text{(unknown)} \tag{2.7}$$

The interior condition

$$\left[\frac{\partial T(r,z)}{\partial z} + T(r,z)\right]\Big|_{z=\zeta} = u(r) \quad \text{(known)} \tag{2.8}$$

The stress functions are

$$\sigma_{rr} = -2\mu \frac{1}{r}\frac{\partial \mathrm{U}}{\partial r} \tag{2.9}$$

$$\sigma_{\theta\theta} = -2\mu \frac{\partial^2 \mathrm{U}}{\partial \mathrm{r}^2} \tag{2.10}$$

where μ : Lame's constant.
In the plane state of stress,
$\sigma_{zz} = \sigma_{rz} = \sigma_{\theta z} = 0$.
The equations (2.1) to (2.10) represent the mathematical formulation of the problem under given boundary conditions.

III. THE INTEGRAL TRANSFORM REQUIRED TO FIND SOLUTION:

The finite Marchi-Zgrablich integral transform of $f(r)$ is defined as

$$f^*{}_p(m) = \int_{r=a}^{r=b} r f(r) S_p\,(h_1,h_2,\xi_m r)dr \tag{3.1}$$

The inversion formula is

$$f(r) = \sum_{m=1}^{\infty} \frac{f^*{}_p(m) S_p\,(h_1,h_2,\xi_m r)}{C_m} \tag{3.2}$$

where

$$C_m = \frac{b^2}{2}\{S_p{}^2(h_1,h_2,\xi_m b) - S_{p-1}(h_1,h_2,\xi_m b).S_{p+1}(h_1,h_2,\xi_m b)\}$$
$$- \frac{a^2}{2}\{S_p{}^2(h_1,h_2,\xi_m a) - S_{p-1}(h_1,h_2,\xi_m a).S_{p+1}(h_1,h_2,\xi_m a)\} \tag{3.3}$$

and the property of this transform is

$$\int_a^b r\left\{\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}-\frac{p^2}{r^2}f\right\}S_p\,(h_1,h_2,\xi_m r)dr$$
$$=\frac{b}{h_2}S_p\,(h_1,h_2,\xi_m b)\left\{f+h_2\frac{\partial f}{\partial r}\right\}_{r=b}-\frac{a}{h_1}S_p\,(h_1,h_2,\xi_m a)\left\{f+h_1\frac{\partial f}{\partial r}\right\}_{r=a}-\xi_m{}^2 f^*{}_p(m) \tag{3.4}$$

where h_1, h_2 are constants in the boundary conditions

$f(r)+h_1 f\ '(r)|_{r=a}=0$

$f(r)+h_2 f\ '(r)|_{r=b}=0$

$f^*{}_p(m)$ is the transform of $f(r)$ with respect to nucleus $S_p\,(h_1,h_2,\xi_m r)$ and weight function r. The eigen values ξ_m are the positive roots of the characteristic equation

$$J_p(h_1\,,\xi a)Y_p(h_2\,,\xi b)-J_p(h_2\,,\xi b)Y_p(h_1\,,\xi a)=0$$

and the kernel function $S_p\,(h_1,h_2,\xi_m r)$ can be defined as

$$S_p\,(h_1,h_2,\xi_m r)=J_p(\,\xi_m r)[\,Y_p(\,h_1,\xi_m a)+Y_p(\,h_2,\xi_m b)]$$
$$-Y_p\,(\,\xi_m r)[\,J_p(\,h_1,\xi_m a)+J_p(\,h_2,\xi_m b)]$$

where $J_p(\,\xi_m r),\ Y_p(\,\xi_m r)$ are Bessel functions of first and second kind respectively.

IV) THE ANALYTICAL SOLUTION IN THE INTEGRAL TRANSFORMS DOMAIN:

4.1) Solution of temperature Distribution:

Application of transform defined in (3.1) to the equation (2.3), (2.6), (2.8), and making use of (2.4) and (2.5),

$$\frac{d^2T^*}{dz^2}(m,z)-\xi_m{}^2\,T^*(m,z)=\Phi(z) \tag{4.1.1}$$

$$\left.\frac{dT^*(m,z)}{dz}\right|_{z=0}=F^*(m) \tag{4.1.2}$$

$$\left.\left(\frac{dT^*(m,z)}{dz}+T^*(m,z)\right)\right|_{z=\zeta}=u^*(m) \tag{4.1.3}$$

where, $\Phi(z)=\frac{a}{h_1}S_0\,(h_1,h_2,\xi_m a)f(z)-\frac{b}{h_2}S_0\,(h_1,h_2,\xi_m b)g(z)$

where, T^*: the Marchi – Zgrablich integral transform of T

m : the integral transform parameter.

The solution of the second order differential Equation (4.1.1) is

$$T^*(m,z)=C_1\cos h\ (\xi_m z)+C_2\sin h\ (\xi_m z)+\text{P. I.} \tag{4.1.4}$$

where , C_1 and C_2 are constants, and particular integral P. I.= $\frac{\Phi(z)}{D^2-\xi_m{}^2}$

Using (4.1.2) and (4.1.3) in (4.1.4) , we get the values of constants C_1 and C_2 as

$$C_1=\frac{u^*(m)-(\text{P.I})_{z=\zeta}-\left[\frac{d}{dz}(\text{P.I})\right]_{z=\zeta}-\left\{\frac{1}{\xi_m}\left[F^*(m)-\left(\frac{\mathrm{d}}{\mathrm{dz}}(\text{P.I})\right)_{z=0}\right][\sin h\ (\xi_m\zeta)+\xi_m\cos h\ (\xi_m\zeta)]\right\}}{\cos h\ (\xi_m\zeta)+\xi_m\sin h\ (\xi_m\zeta)} \tag{4.1.5}$$

$$C_2=\frac{1}{\xi_m}\left[F^*(m)-\left(\frac{\mathrm{d}}{\mathrm{dz}}(\text{P.I})\right)_{z=0}\right] \tag{4.1.6}$$

Substituting values of C_1 and C_2 from (4.1.5) and (4.1.6) in (4.1.4), we have

$$T^*(m, z) = \left\{ \begin{array}{c} \left[u^*(m) - (\mathrm{P.I})_{z=\zeta} - \left[\frac{d}{dz}(\mathrm{P.I})\right]_{z=\zeta}\right] \times \left[\frac{\cos h\ (\xi_m z)}{[\cos h\ (\xi_m \zeta) + \xi_m \sin h\ (\xi_m \zeta)]}\right] \\ + \left\{\left[\frac{1}{\xi_m}\left[F^*(m) - \left(\frac{\mathrm{d}}{\mathrm{dz}}(\mathrm{P.I})\right)_{z=0}\right]\right] \times \left[\frac{[\sin h\ (\xi_m(z-\zeta)) - \xi_m \cos h\ (\xi_m(z-\zeta))]}{[\cos h\ (\xi_m \zeta) + \xi_m \sin h\ (\xi_m \zeta)]}\right]\right\} \\ +P.I. \end{array} \right\} \tag{4.1.7}$$

Applying (3.2) to (4.1.7), we get,

$$T(r,z) = \sum_{m=1}^{\infty} \frac{1}{C_m} \left\{ \begin{array}{c} \left[u^*(m) - (\mathrm{P.I})_{z=\zeta} - \left[\frac{d}{dz}(\mathrm{P.I})\right]_{z=\zeta}\right] \times \left[\frac{\cos h\ (\xi_m z)}{[\cos h\ (\xi_m \zeta) + \xi_m \sin h\ (\xi_m \zeta)]}\right] \\ + \left\{\left[\frac{1}{\xi_m}\left[F^*(m) - \left(\frac{\mathrm{d}}{\mathrm{dz}}(\mathrm{P.I})\right)_{z=0}\right]\right] \times \left[\frac{[\sin h\ (\xi_m(z-\zeta)) - \xi_m \cos h\ (\xi_m(z-\zeta))]}{[\cos h\ (\xi_m \zeta) + \xi_m \sin h\ (\xi_m \zeta)]}\right]\right\} \\ +P.I. \end{array} \right\} \times S_0(h_1, h_2, \xi_m r) \tag{4.1.8}$$

This equation (4.1.8) gives the required temperature distribution .
Also, from (2.7) and (4.1.8),

$$G(r) = \sum_{m=1}^{\infty} \frac{1}{C_m} \left\{ \begin{array}{c} \left[u^*(m) - (\mathrm{P.I})_{z=\zeta} - \left[\frac{d}{dz}(\mathrm{P.I})\right]_{z=\zeta}\right] \times \left[\frac{\cos h\ (\xi_m h)}{[\cos h\ (\xi_m \zeta) + \xi_m \sin h\ (\xi_m \zeta)]}\right] \\ + \left\{\left[\frac{1}{\xi_m}\left[F^*(m) - \left(\frac{\mathrm{d}}{\mathrm{dz}}(\mathrm{P.I})\right)_{z=0}\right]\right] \times \left[\frac{[\sin h\ (\xi_m(h-\zeta)) - \xi_m \cos h\ (\xi_m(h-\zeta))]}{[\cos h\ (\xi_m \zeta) + \xi_m \sin h\ (\xi_m \zeta)]}\right]\right\} \\ +P.I. \end{array} \right\} \times S_0(h_1, h_2, \xi_m r) \tag{4.1.9}$$

4.2) Determination of displacement function :

Substituting the value of $T(r,z)$ from equation (4.1.8) in equation (2.1), we have

$$U(r,z) = -(1+\nu)\alpha \times \sum_{m=1}^{\infty} \frac{1}{C_m {\xi_m}^2} \left\{ \begin{array}{c} \left[u^*(m) - (\mathrm{P.I})_{z=\zeta} - \left[\frac{d}{dz}(\mathrm{P.I})\right]_{z=\zeta}\right] \times \left[\frac{\cos h\ (\xi_m z)}{[\cos h\ (\xi_m \zeta) + \xi_m \sin h\ (\xi_m \zeta)]}\right] \\ + \left\{\left[\frac{1}{\xi_m}\left[F^*(m) - \left(\frac{\mathrm{d}}{\mathrm{dz}}(\mathrm{P.I})\right)_{z=0}\right]\right] \times \left[\frac{[\sin h\ (\xi_m(z-\zeta)) - \xi_m \cos h\ (\xi_m(z-\zeta))]}{[\cos h\ (\xi_m \zeta) + \xi_m \sin h\ (\xi_m \zeta)]}\right]\right\} \\ +P.I. \end{array} \right\} \times S_0(h_1, h_2, \xi_m r) \tag{4.2.1}$$

4.3) Determination of stress components :

Substituting eq. (4.2.1) in eq. (2.8), (2.9), we have

$$\sigma_{rr} = \frac{2\mu(1+\nu)\alpha}{r} \times \sum_{m=1}^{\infty} \frac{1}{C_m \xi_m{}^2} \left\{ \begin{array}{l} \left[u^*(m) - (\mathrm{P.I})_{z=\zeta} - \left[\frac{d}{dz}(\mathrm{P.I}) \right]_{z=\zeta} \right] \times \left[\frac{\cos h\,(\xi_m z)}{[\cos h\,(\xi_m \zeta) + \xi_m \sin h\,(\xi_m \zeta)]} \right] \\ + \left\{ \left[\frac{1}{\xi_m} \left[F^*(m) - \left(\frac{\mathrm{d}}{\mathrm{dz}}(\mathrm{P.I}) \right)_{z=0} \right] \right] \times \left[\frac{[\sin h\,(\xi_m(z-\zeta)) - \xi_m \cos h\,(\xi_m(z-\zeta))]}{[\cos h\,(\xi_m \zeta) + \xi_m \sin h\,(\xi_m \zeta)]} \right] \right\} \\ +P.I. \end{array} \right\} \times S_0{}'(h_1, h_2, \xi_m r) \tag{4.3.1}$$

$$\sigma_{\theta\theta} = 2\mu(1+\nu)\alpha \times \sum_{m=1}^{\infty} \frac{1}{C_m \xi_m} \left\{ \begin{array}{l} \left[u^*(m) - (\mathrm{P.I})_{z=\zeta} - \left[\frac{d}{dz}(\mathrm{P.I}) \right]_{z=\zeta} \right] \times \left[\frac{\cos h\,(\xi_m z)}{[\cos h\,(\xi_m \zeta) + \xi_m \sin h\,(\xi_m \zeta)]} \right] \\ + \left\{ \left[\frac{1}{\xi_m} \left[F^*(m) - \left(\frac{\mathrm{d}}{\mathrm{dz}}(\mathrm{P.I}) \right)_{z=0} \right] \right] \times \left[\frac{[\sin h\,(\xi_m(z-\zeta)) - \xi_m \cos h\,(\xi_m(z-\zeta))]}{[\cos h\,(\xi_m \zeta) + \xi_m \sin h\,(\xi_m \zeta)]} \right] \right\} \\ +P.I. \end{array} \right\} \times S_0{}''(h_1, h_2, \xi_m r) \tag{4.3.2}$$

4.4) Special case and Numerical results:

Set $F(r) = \delta(r-1.5)$ and $u(r) = \delta(r-1.5)e^{\zeta}$, $f(z) = g(z) = z,\ a = 1, b = 2,$ $h = 0.2, h_1 = h_2 = 0.25, \zeta = 0.15$

in the equation (4.1.9) , we have,

$$G(r) = \sum_{m=1}^{\infty} \frac{1}{C_m} \left\{ \begin{array}{l} \left[(1.74) S_0(0.25, 0.25, 1.5\xi_m) + \frac{0.15N}{\xi_m{}^2} \right] \times \left[\frac{\sin h\,(0.2\xi_m)}{[\cos h\,(0.15\xi_m) + \xi_m \sin h\,(0.15\xi_m)]} \right] \\ + \left\{ [(1.5) S_0(0.25, 0.25, 1.5\xi_m)] \times \left[\frac{[\sin h\,(0.05\xi_m) - \xi_m \cos h\,(0.05\xi_m)]}{[\cos h\,(0.15\xi_m) + \xi_m \sin h\,(0.15\xi_m)]} \right] \right\} \\ - \frac{0.2N}{\xi_m{}^2} \end{array} \right\} \times S_0(0.25, 0.25, \xi_m r) \tag{4.4.1}$$

V. CONCLUSION

In the present paper , analysis of Temperature distribution , displacement function and associated thermal stresses in a thin annular disc is done . The inverse , steady state thermoelastic problem is discussed and solved by using Marchi Zgrablich transform technique . The results are obtained in terms of Bessel's function in the form of infinite series. Considering a special case, unknown temperature is obtained.

By assigning required values of the parameters and functions in the obtained expressions, any particular case useful in applications can be studied. The results obtained here will find its applications in the engineering issues in areas where the thermal analysis of disc is necessary.

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